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Partial generalized crossed products and a seven term exact sequence ☆

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Given a non-necessarily commutative unital ring R and a unital partial representation Θ of a group G into the Picard semigroup $\mathbf{PicS}(R)$ of the isomorphism classes of partially invertible R-bimodules, we construct an abelian group $\mathcal{C}(\Theta/R)$ formed by the isomorphism classes of partial generalized crossed products related to Θ and identify an appropriate second partial cohomology group of G with a naturally defined subgroup $\mathcal{C}_0(\Theta/R)$ of $\mathcal{C}(\Theta/R)$. Then we use the obtained results to give an analogue of the Chase-Harrison-Rosenberg exact sequence associated with an extension of non-necessarily commutative rings $R \subseteq S$ with the same unity and a unital partial representation $G \to \mathcal{S}_R(S)$ of an arbitrary group G into the monoid $\mathcal{S}_R(S)$ of the R-subbimodules of S. This generalizes the works by Kanzaki and Miyashita.

1. Introduction

The purpose of this paper is to give a partial action version of the Y. Miyashita's non-commutative analogue [45] of the Chase-Harrison-Rosenberg sequence [7]. The latter is related to a Galois extension $R^G \subseteq R$ of commutative unital rings with a finite Galois group G and was obtained applying a seven term Amitsur cohomology exact sequence of S. U. Chase and A. Rosenberg [8], proved using spectral sequences. A constructive proof of the Chase-Harrison-Rosenberg sequence was given by T. Kanzaki [37], employing the novel notion of a generalized crossed product. The sequence is of the form

$$0 \to H^1(G,\mathcal{U}(R))\to\mathbf{Pic}(R^G)\to\mathbf{Pic}(R)^G\to H^2(G,\mathcal{U}(R))\to B(R/R^G)\to H^1(G,\mathbf{Pic}(R))\to H^3(G,\mathcal{U}(R)),$$

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where $\mathbf{Pic}(R)$ is the Picard group of the isomorphism classes of finitely generated projective R-modules of rank 1, $H^n(G,\_\ )$ is the cohomology group of G and $B(R/R^G)$ is the relative Brauer group, whose elements are the equivalence classes of the Azumaya R^G-algebras split by R. There is an induced action of G on $\mathbf{Pic}(R)$, and $\mathbf{Pic}(R)^G$ denotes the subgroup of the elements fixed by the action. If R is a field, then the first homomorphism of the sequence immediately gives Emmy Noether's theorem $H^1(G,\mathcal{U}(R)) = 1$, which extends the E. Kummer's result of 1855 on cyclic field extensions, known as Hilbert's Theorem 90 due to the fact of being listed under number 90 in D. Hilbert's Zahlbericht. Another important fact, being generalized by the sequence, is the crossed product theorem, stating that if R is a field, then the map $H^2(G,\mathcal{U}(R)) \ni [f] \mapsto [R *_f G] \in B(R/R^G)$ is an isomorphism of groups, where $R *_f G$ stands for the crossed product associated with the 2-cocycle f and the Galois action of G on R.

Besides Galois theory, crossed products are relevant in ring theory, in the theory of von Neumann algebras and in that of C^*-algebras. In the latter area more general types of crossed products were introduced, in particular, crossed products by partial actions, which are useful to deal with important classes of C^*-algebras generated by partial isometries. Prominent early examples of partial crossed product descriptions include those for the Cuntz-Krieger C^*-algebras [29] and for the more general Exel-Laca C^*-algebras [33]. Among the more recent relevant examples we mention the tame C^*-algebras related to finite bipartite separated graphs [1], the full and reduced C^*-algebras of E-unitary or strongly E^*-unitary inverse semigroups [43], the Carlsen-Matsumoto C^*-algebras associated to arbitrary subshifts on finite alphabets [16] and certain ultragraph C^*-algebras [35]. Highly interesting particular cases include C^*-algebras associated to dynamical systems of type (m, n) (see [2]) and graph C^*-algebras (see [32]).

In algebra, partial crossed products turned out be applicable to group graded algebras [15], [18], to Hecke algebras [31], to Leavitt path algebras [34] and to Steinberg algebras [9], [5], [36]. In particular, it was shown in [18] that any group graded algebra A, satisfying a natural mild condition, after passing to finite matrices over A of an appropriate infinite size, becomes a crossed product by a twisted partial action. This is an algebraic analogue of R. Exel's result [30] on C^*-algebraic bundles. In addition, partial crossed product analogues are useful for semigroups, as it can be seen in [38], [10], [40], [41], [42].

Among the diverse algebraic developments on partial actions we mention two steps related to our theme: the partial action treatment of Galois theory of commutative rings in [19] and the introduction of a group cohomology based on partial actions in [20]. While the former inspired a Hopf theoretic approach to partial actions by S. Caenepeel and K. Janssen in [6], which in its turn became a starting point for interesting and fruitful Hopf theoretic developments, the latter is useful for partial projective group representations [25], for extensions of semi-lattices of groups by groups [21] and to the study of the ideal structure of reduced (global) C^*-crossed products [39]. It also influenced the introduction of partial cohomology of Hopf algebras in [4] and that of groupoids in [48]. For more information on advances around partial actions and applications the reader is referred to R. Exel's book [32] and to the survey article [14].

In view of the above mentioned progresses, it was natural to generalize the Chase-Harrison-Rosenberg sequence to partial Galois extensions of commutative rings. This was done in a sequence of two papers [22] and [23]. New ingredients came into the picture: the first one is the inverse semigroup $\mathbf{PicS}(R)$ of the finitely generated projective R-modules of rank ≤ 1, on which the Galois partial action α of the finite group G on the commutative unital ring R induces a partial action α^*; the second one is certain partial representations needed to introduce a partial version of generalized crossed products. All (global) cohomology groups in the Chase-Harrison-Rosenberg sequence are substituted by their partial theoretic analogues, and the sequence takes the form:

$$0 \to H^1(G, \alpha, R) \to \mathbf{Pic}(R^\alpha) \to \mathbf{PicS}(R)^{\alpha^*} \cap \mathbf{Pic}(R) \to H^2(G, \alpha, R) \to B(R/R^\alpha) \to$$
$$H^1(G, \alpha^*, \mathbf{PicS}(R)) \to H^3(G, \alpha, R),$$

where R^{α} and $\mathbf{PicS}(R)^{\alpha^*}$ are the subring and the submonoid of partial invariants, respectively (see Section 3.1 for definitions). Two immediate consequences of the sequence are a partial version of the Hilbert's Theorem 90 [23, Corollary 6.8] and the partial crossed product theorem [23, Corollary 6.9].

T. Kanzaki's proof of the Chase-Harrison-Rosenberg sequence inspired Y. Miyashita to use generalized crossed products to produce a non-commutative analogue of the sequence [45]. Instead of a Galois extension, Miyashita's sequence is related to a rather general setting of a ring extension $R \subseteq S$ with the same unity and a fixed homomorphism $\Theta : G \to \mathbf{Inv}_R(S)$, where G is an arbitrary non-necessarily finite group and $\mathbf{Inv}_R(S)$ is the group of invertible R-subbimodules of S. The sequence has a somewhat different form, but the case of a Galois extension of commutative rings $R^G \subseteq R$ can be recovered by taking $S = R \star G$ and a rather natural homomorphism $G \to \mathbf{Inv}_R(R \star G)$, where $R * G$ is the skew group ring by the Galois action of G on R. One of the crucial ingredients of the non-commutative version of the sequence is an appropriate analogue $\mathcal{B}(\Theta/R)$ of the Brauer group. It is defined as a quotient of the abelian group $\mathcal{C}(\Theta/R)$ of the isomorphism classes of certain generalized crossed products related to Θ. A subgroup $\mathcal{C}_0(\Theta/R)$ of $\mathcal{C}(\Theta/R)$ is shown to be isomorphic to a second cohomology group of G, and the restriction of the natural map $\mathcal{C}(\Theta/R) \to \mathcal{B}(\Theta/R)$ to $\mathcal{C}_0(\Theta/R)$ provides one of the homomorphisms of the sequence. Miyashita's sequence was extended for a technically more difficult case of rings with local units by L. El Kaoutit and J. Gómez-Torrecillas in [28] using their results from [27].

Several other articles related to the Chase-Harrison-Rosenberg sequence were published. In particular, in [12] D. Crocker, I. Raeburn and D. Williams obtained a C^*-analogue of the sequence, using the equivariant version of the Picard group [12, Proposition 2], that of the Brauer group [11] and the cohomology theory developed by C. C. Moore [46] for actions of locally compact groups on Polish modules.

We begin by giving some preliminary facts on projective modules over a non-necessarily commutative ring R with 1 in Section 2.1, followed by the notion of the generalized Picard semigroup $\mathbf{PicS}(R)$ of the isomorphism classes of the partially invertible R-bimodules in Section 2.2 and by some technical facts on the bimodule relation $M|N$ and isomorphisms of the form $M \otimes_R N \simeq N \otimes_R M$ in Section 2.3. Furthermore, given an extension of rings $R \subseteq S$ with the same unity, we recall in Section 2.4 the definition of the group $\mathcal{P}(S/R)$ of the isomorphism classes of certain R-bimodule maps $P \to X$, where P is an R-bimodule, whose isomorphism class $[P]$ lies in the Picard group $\mathbf{Pic}(R)$ and $[X] \in \mathbf{Pic}(S)$. The group $\mathcal{P}(S/R)$ participates an exact sequence given in [45, Theorem 1.5] (see also [27, Proposition 5.1]), which is used to construct the first two homomorphisms of our main sequence, whose second term is an appropriate subgroup of $\mathcal{P}(S/R)$.

Section 3 deals with partial generalized crossed products. After recalling in Section 3.1 some background on partial actions, partial representations and partial cohomology of a group G, we give in Section 3.2 some auxiliary facts related to a unital partial representation of the form $G \to \mathbf{PicS}(R)$ and, given such a representation, construct the associated partial actions α and α^* on $\mathcal{Z}$ and $\mathbf{PicS}(R)$, respectively, where $\mathcal{Z}$ stands for the center of R. In particular, this allows us to consider partial cohomology groups of a group G with values in $\mathcal{Z}$, which we denote by $H^n_{\Theta}(G, \alpha, \mathcal{Z})$. Notice that along with the usual concept of a partial action of a group on a semigroup, in which the domains of the partial isomorphisms are assumed to be two-sided ideals, it may be useful to keep in mind a relaxed version allowing them to be only subsemigroups (see Definition 3.1 and Proposition 3.3).

The main technical part of Section 3 is concentrated in Section 3.3, where we give the concept of a factor set and that of a generalized crossed product related to a fixed unital partial representation $\Theta : G \to \mathbf{PicS}(R)$, and prove several results on them. Factor sets are needed to produce generalized partial crossed products and they may arise from partial representations Θ as families of R-bimodule isomorphisms of the form $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$, where 1_x, $(x \in G)$, is a central idempotent in R. An obstruction for f^{Θ} to be a factor set is a partial 3-cocycle (see Proposition 3.15 and Corollary 3.16), which is automatically trivial if the partial representation comes from that of the form $\Theta : G \to \mathcal{S}_R(S)$. Here $\mathcal{S}_R(S)$ denotes the set of the R-subbimodules of S, equipped with the natural multiplication $MN = \left\{\sum m_i n_i;\ m_i \in M, n_i \in N\right\}$, $M, N \in \mathcal{S}_R(S)$, turning $\mathcal{S}_R(S)$ into a monoid with neutral element R. Section 3.4 gives some preliminaries on unital partial representations $G \to \mathcal{S}_R(S)$.

As it can be seen already in [22] and [23], it is more laborious to deal with the partial action setting. The non-commutative case brings new technical challenges, and a big part of the work is related to the partial action version of the group $\mathcal{C}(\Theta/R)$, to which Section 4 is dedicated. More specifically, given a unital partial representation $\Theta : G \longrightarrow \mathbf{PicS}(R)$ we construct in Theorem 4.1 the abelian group $\mathcal{C}(\Theta/R)$ of the isomorphism classes of certain generalized crossed products related to Θ. Its subgroup $\mathcal{C}_0(\Theta/R)$ is given in Proposition 4.2 and an isomorphism between $\mathcal{C}_0(\Theta/R)$ and a second partial cohomology group of G is established in Theorem 4.4.

Our seven term exact sequence is obtained in a series of subsections of Section 5, and it is produced starting with an extension of rings $R \subseteq S$ with the same unity and a fixed unital partial representation $\Theta : G \to \mathcal{S}_R(S)$. An important step is the construction in Section 5.2 of a group homomorphism $\mathcal{L} : \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} \to \mathcal{C}(\Theta/R)$ (see Theorem 5.10), where $\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ is a subgroup of $\mathbf{Pic}_{\mathcal{Z}}(R) \subseteq \mathbf{Pic}(R)$ (see Section 2.2 and Lemma 5.6). The homomorphism $\mathcal{L}$ is used to define our analogue of the Brauer group in Section 5.3 as the quotient $\mathcal{B}(\Theta/R) = \dfrac{\mathcal{C}(\Theta/R)}{\mathrm{Im}(\mathcal{L})}$.

The sixth term $\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ of our sequence is introduced in Section 5.4 and it involves 1-cocycles with values in the subsemigroup $\mathbf{PicS}_0(R) \subseteq \mathbf{PicS}(R)$ provided by Lemma 5.20. The same lemma implies that the partial action α^* of G on $\mathbf{PicS}(R)$ can be restricted to $\mathbf{PicS}_0(R)$. Then the group $\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ is defined as the quotient $\dfrac{Z^1(G, \alpha^*, \mathbf{PicS}_0(R))}{(\zeta \circ \mathcal{L})(\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)})}$, where $\zeta : \mathcal{C}(\Theta/R) \longrightarrow Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$ is the group homomorphism given in Lemma 5.21. It follows by Remark 5.22 that $\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ is a quotient of the partial cohomology group $H^1(G, \alpha^*, \mathbf{PicS}_0(R))$.

Besides being used in the definition of our version $\mathcal{B}(\Theta/R)$ of a Brauer group, the map $\mathcal{L}$ is involved in the construction of three consecutive homomorphisms of our sequence:

$$\mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \to H^2_\Theta(G, \alpha, \mathcal{Z}) \to \mathcal{B}(\Theta/R) \to \overline{H}^1(G, \alpha^*, \mathbf{PicS}_0(R))$$

(see Proposition 5.19, Proposition 5.23 and the definition of φ_3 after the proof of Theorem 5.10). The entire sequence is given in Theorem 5.27, and it is of the form:

$$\begin{aligned} 1 &\to H^1_\Theta(G, \alpha, \mathcal{Z}) \to \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)} \to \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \to H^2_\Theta(G, \alpha, \mathcal{Z}) \to \mathcal{B}(\Theta/R) \\ &\to \overline{H}^1(G, \alpha^*, \mathbf{PicS}_0(R)) \to H^3_\Theta(G, \alpha, \mathcal{Z}), \end{aligned}$$

where the subgroup $\mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)}$ of $\mathcal{P}(\Delta(\Theta)/R)$ is defined in Section 5.1 (see Lemma 5.2).

We omit some routine technical verifications. An insistent reader should be able to recover them. Alternatively, they may be found in the expanded version of the paper available in arXiv [24].

2. Background

In all what follows R will stand for an associative ring with unity element 1. We denote by $\mathcal{Z}$ the center of R and by $\mathcal{U}(R)$ the unit group of R. A right R-module M is called unital if $m \cdot 1 = m$, for all $m \in M$. In this case, we have that $MR = R$ and $M \otimes_R R \simeq M$ via the right R-action on M, i.e. $m \otimes r \mapsto m \cdot r$. Analogously we define a left unital R-module. We shall sometimes write M_R to say that M is a right module over R. The meaning of ${}_RM$ will be similar. Given an R-bimodule M we denote by $\mathbf{Aut}_{R-R}(M)$ the group of all (R, R)-bimodule automorphisms of M. Moreover, the bimodule M is called central if $r \cdot m = m \cdot r$, for all $m \in M$ and $r \in R$.

Let e be a central idempotent of R and M an R-bimodule. Write $eM = \{m \in M : em = m\}$. Obviously, eM is an R-subbimodule of M, since e is a central element. Moreover, there is an R-bimodule isomorphism $Re \otimes_R M \simeq eM$ defined by $r \otimes m \mapsto rm$, whose inverse is given by $m \mapsto e \otimes m$. Analogously, $Me = \{m \in M : me = m\}$ is a R-subbimodule of M and $M \otimes_R Re \simeq Me$, as R-bimodules. If $F : M \to N$ is an isomorphism of R-bimodules, then the restriction of F to eM is an R-bimodule isomorphism onto eN. In some cases, for simplicity of notation, we will denote this restriction by the same symbol F.

2.1. Projective modules over non-commutative rings

In this section we list some properties of finitely generated projective R-modules, whose straightforward proofs are left to the reader.

Let M and N be R-bimodules. Then the sets $\mathrm{Hom}(M_R, N_R)$ and $\mathrm{Hom}({}_RM, {}_RN)$ are R-bimodules via:

$$(r \cdot f)(m) = rf(m) \quad \text{and} \quad (f \cdot r)(m) = f(rm), \quad f \in \mathrm{Hom}(M_R, N_R), m \in M, r \in R, \tag{1}$$

$$(r \cdot g)(m) = g(mr) \quad \text{and} \quad (g \cdot r)(m) = g(m)r, \quad g \in \mathrm{Hom}({}_RM, {}_RN), m \in M, r \in R. \tag{2}$$

In particular, taking $N = R$, we have that the sets ${}^*M = \mathrm{Hom}(M_R, R_R)$ and $M^* = \mathrm{Hom}({}_RM, {}_RR)$ are R-bimodules via (1) and (2).

Lemma 2.1. *Let P be an R-bimodule.*

(i) *If P is finitely generated projective as a right R-module, then the map $\varphi : P \otimes_R {}^*P \longrightarrow End(P_R)$, defined by $\varphi(p \otimes f)(p') = pf(p')$, is an R-bimodule isomorphism.*
(ii) *If P is finitely generated projective as a left R-module, then the map $\varphi' : P^* \otimes_R P \longrightarrow End({}_RP)$, defined by $\varphi(f \otimes p)(p') = f(p')p$, is an R-bimodule isomorphism.*

Lemma 2.2. *Let P and Q be R-bimodules. If P and Q are finitely generated projective as right (left) R-modules, then $P \otimes_R Q$ is a right (left) finitely generated projective R-module.*

Lemma 2.3. *Let P and Q be R-bimodules.*

(i) *If P and Q are finitely generated projective as right R-modules, then the map $\eta : {}^*P \otimes_R {}^*Q \to {}^*(Q \otimes_R P)$, defined by $\eta(f \otimes g)(p \otimes q) = f(g(q)p)$, is an R-bimodule isomorphism.*
(ii) *If P and Q are finitely generated projective as left R-modules, then the map $\eta' : P^* \otimes_R Q^* \to (Q \otimes_R P)^*$, defined by $\eta'(f \otimes g)(q \otimes p) = g(qf(p))$, is an R-bimodule isomorphism.*

2.2. The generalized Picard semigroup

We recall that an R-bimodule P is called *invertible* if there exists an R-bimodule Q such that

$$P \otimes_R Q \simeq R \simeq Q \otimes_R P,$$

as R-bimodules (see, for example, [3, Chapter II]). We shall need the following known fact.

Proposition 2.4. *[3, Chapter II] Let P be an R-bimodule. Then the following are equivalent:*

(a) *P is invertible.*
(b) *P is a finitely generated projective right R-module, which is a generator and $R \simeq End(P_R)$.*
(c) *P is a finitely generated projective left R-module, which is a generator and $R \simeq End({}_RP)$.*

We also recall that the *Picard group* $\mathbf{Pic}(R)$ of a ring R is the set of the isomorphism classes of invertible R-bimodules with multiplication given by $\otimes_R$. By [47, Theorem 1.1], ${}^*P \simeq P^*$ as R-bimodules, and hence $[{}^*P] = [P^*] = [P]^{-1}$ in $\mathbf{Pic}(R)$. We shall write $[P]^{-1} = [P^{-1}]$. Note that we can choose the isomorphisms $\mathfrak{r}$ and $\mathfrak{l}$ so that $(R, R, P, P^*, \mathfrak{r}, \mathfrak{l})$ is a Morita context, and consequently:

$$\mathfrak{l}(p \otimes q)p' = p\mathfrak{r}(q \otimes p') \quad \text{and} \quad \mathfrak{r}(q \otimes p)q' = q\mathfrak{l}(p \otimes q'), \tag{3}$$

for all $p, p' \in P$ and $q, q' \in P^*$ (see [3, Chapter II]). For a K-algebra R over a commutative ring K we denote by $\mathbf{Pic}_K(R)$ the set of the isomorphism classes of invertible R-bimodules which are central over K, i.e., $kp = pk$ for all $k \in K$ and $p \in P$. In the case when R is commutative, we have that $[P] \in \mathbf{Pic}_R(R)$ if and only if P is a finitely generated projective R-module of rank 1 (see, for example, [13, II, §5]).[1]

We say that an R-bimodule P is *partially invertible* if

(i) P is finitely generated projective left and right R-module;
(ii) The maps

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{End}(P_R) \\ r & \longmapsto & (p \mapsto rp) \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \longrightarrow & \mathrm{End}({}_RP) \\ r & \longmapsto & (p \mapsto pr) \end{array}$$

are epimorphisms.

Example 2.5. Let $e \in R$ be a central idempotent and P an invertible Re-bimodule. Define a structure of an R-bimodule on P by setting $r \cdot p = (er)p$ and $p \cdot r = p(er), r \in R, p \in P$. Then $\mathrm{End}(P_R) = \mathrm{End}(P_{eR}), \mathrm{End}({}_RP) = \mathrm{End}({}_{eR}P)$ and it is readily seen that P is a partially invertible R-bimodule.

We denote by $\mathbf{PicS}(R)$ the set of the isomorphism classes $[P]$ of partially invertible R-bimodules, that is,

$$\mathbf{PicS}(R) = \{[P], P \text{ is a partially invertible } R\text{-bimodule}\}.$$

Proposition 2.6. $\mathbf{PicS}(R)$ *is a monoid with multiplication defined by* $[P][Q] = [P \otimes_R Q]$.

Proof. By Lemma 2.2, we have that $P \otimes_R Q$ is a left and right finitely generated projective R-module. Let ψ be the map defined by the following chain of R-bimodule isomorphisms:

$$\begin{tikzcd}
R \arrow[r] \arrow[ddddrrrr, dashed, bend right, "\psi"'] & \mathrm{End}(P_R) \arrow[r, "\varphi_P^{-1}"] & P \otimes_R {}^*P \arrow[r] & P \otimes_R R \otimes_R {}^*P \arrow[r] & P \otimes_R \mathrm{End}(Q_R) \otimes_R {}^*P \arrow[d, "P \otimes_R \varphi_Q^{-1} \otimes_R {}^*P"] \\
 & & & & P \otimes_R Q \otimes_R {}^*Q \otimes_R {}^*P \arrow[d, "P \otimes Q \otimes \eta"] \\
 & & & & P \otimes_R Q \otimes_R {}^*(P \otimes_R Q) \arrow[d, "\varphi_{P \otimes Q}"] \\
 & & & & \\
 & & & & \mathrm{End}(P \otimes_R Q)
\end{tikzcd}$$

[1] If R is commutative, then $\mathbf{Pic}_R(R)$ is usually denoted simply by $\mathbf{Pic}(R)$.

where φ_P, φ_Q and $\varphi_{P\otimes_R Q}$ are as in Lemma 2.1 and η is the isomorphism given by Lemma 2.3. Then ψ is onto and it is directly verified that

$$\psi(r)(p \otimes q) = rp \otimes q.$$

Analogously, $\psi' : R \longrightarrow \mathrm{End}({}_R P \otimes_R Q)$, determined by $\psi'(r)(p \otimes q) = p \otimes qr$, is onto. Thus, $[P\otimes_R Q] \in \mathbf{PicS}(R)$. Clearly, $[R]$ is the unity element of $\mathbf{PicS}(R)$. Therefore, $\mathbf{PicS}(R)$ is a monoid. $\square$

Remark 2.7. It is immediately seen that $\mathcal{U}(\mathbf{PicS}(R)) = \mathbf{Pic}(R)$.

With respect to Example 2.5 note that if the eR-bimodule Q is an inverse for the eR-bimodule P, then considering P and Q as R-bimodules it is easily seen that $[P][Q][P] = [P]$ and $[Q][P][Q] = [Q]$ in $\mathbf{PicS}(R)$, so that the R-bimodule Q is a partial (or weak) inverse of the R-bimodule P.

Analogously to the case of the Picard groups, if R is a K-algebra, then $\mathbf{PicS}_K(R)$ stands for the subsemigroup of those isomorphism classes $[P] \in \mathbf{PicS}(R)$, in which the bimodules P are central over K. If R is a commutative ring, then $\mathbf{PicS}(R) = \mathbf{PicS}_R(R)$ consists of the isomorphism classes of the finitely generated projective central R-bimodules of rank less than or equal to one (see [22, Proposition 3.6]). Moreover, by [22, Proposition 3.8], we have that $\mathbf{PicS}(R)$ is an inverse semigroup, with $[P]^* = [P^*]$ for all $[P] \in \mathbf{PicS}(R)$.

2.3. The bimodule relation $M|N$ and a tensor product commuting isomorphism

The central idempotents of the ring R will be important for us. We shall use the following easy isomorphism. Let e be a central idempotent in R and M an R-bimodule. Write $eM = \{m \in M; em = m\}$. Since e is central in R, then eM is an R-subbimodule of M. Moreover, we have the following isomorphism of R-bimodules:

$$\begin{array}{rcl} Re \otimes_R M & \longrightarrow & eM, \\ r \otimes m & \longmapsto & rm, \end{array} \tag{4}$$

whose inverse is given by $eM \ni m \longmapsto e \otimes m \in Re \otimes_R M$. Analogously, $Me = \{m \in M;\ me = m\}$ is an R-subbimodule of M and we have the R-bimodule isomorphism $M \otimes_R Re \simeq Me$ defined by $m \otimes re \longmapsto mre$. Let M and N be R-bimodules, $F : M \longrightarrow N$ an R-bimodule isomorphism and e a central idempotent in R. The restriction of F to the R-subbimodule eM is also an R-bimodule isomorphism between eM and eN, which, with a slight abuse of notation, will be denoted by the same symbol F.

Let M and N be R-bimodules. We shall write $M|N$ if M is isomorphic, as an R-bimodule, to a direct summand of some direct power of N, that is, if there exists an R-bimodule M' such that $N^{(n)} \simeq M \oplus M'$, for some $n \in \mathbb{N}$. This is a reflexive and transitive relation which is compatible with the tensor product, in the sense that, if $M|N$ and Q is an R-bimodule, then

$$(M \otimes_R Q)|(N \otimes_R Q) \ \text{ and } \ (Q \otimes_R M)|(Q \otimes_R N). \tag{5}$$

Remark 2.8. The first two of the following facts are immediate, whereas the verification of the third one is straightforward.

(i) If $M_R|R_R$, then M is a finitely generated projective right R-module.
(ii) If $R_R|M_R$, then M_R is a generator of the category of the right R-modules.
(iii) $M|N$ if and only if there exist R-bimodule homomorphisms $f_i : M \to N$ and $g_i : N \to M$, $i = 1, 2, ..., n$, such that $\sum_{i=1}^n g_i \circ f_i = Id_M$.

Remark 2.9. If M is an R-bimodule such that $M|R$, then M is a central $\mathcal{Z}$-bimodule. Indeed, there exist R-bilinear maps $f_i : M \longrightarrow R$ and $g_i : R \longrightarrow M$, $i = 1, 2, ..., n$, with $\sum_{i=1}^{n} g_i f_i = Id_M$. Take $m \in M$ and $r \in \mathcal{Z}$, then

$$mr = \sum_{i=1}^{n} g_i(f_i(m))r = \sum_{i=1}^{n} g_i(\underbrace{f_i(m)}_{\in R} r) = \sum_{i=1}^{n} g_i(r f_i(m)) = \sum_{i=1}^{n} r g_i(f_i(m)) = rm.$$

Hence, M is central over $\mathcal{Z}$.

Proposition 2.10. *[45, Corollary 3] Let M and N be R-bimodules. If $M|R$ and $N|R$, then there is an R-bimodule isomorphism $M \otimes_R N \simeq N \otimes_R M$ given by*

$$\begin{array}{rccc} T_{M,N}: & M \otimes_R N & \longrightarrow & N \otimes_R M \\ & x \otimes y & \longmapsto & \sum_{i=1}^{n} f_i(x) y \otimes g_i(1), \end{array}$$

where $f_i : M \to R$, $g_i : R \to M$, for $i = 1, 2, ..., n$, are R-bilinear maps, such that $\sum_{i=1}^{n} g_i f_i = Id_M$, as given by Remark 2.8.

Given an R-bimodule M, we write $C_M(R) = \{m \in M,\ rm = mr \text{ for all } r \in R\}$. The abelian group $C_M(R)$ is a central $\mathcal{Z}$-bimodule, and we may endow $R \otimes_{\mathcal{Z}} C_M(R)$ with a structure of an R-bimodule via the R-bimodule structure of R, that is,

$$r_1 \cdot (r \otimes_{\mathcal{Z}} m) \cdot r_2 = r_1 r r_2 \otimes_{\mathcal{Z}} m,$$

for all $r, r_1, r_2 \in R$ and $m \in C_M(R)$.

Lemma 2.11. *[45, Lemma 2.4] Let M be an R-bimodule such that $M|R$ (as bimodules), then there exists an R-bimodule isomorphism $M \simeq R \otimes_{\mathcal{Z}} C_M(R)$.*

Corollary 2.12. *Let M and N be R-bimodules such that $M|R$ and N is central over $\mathcal{Z}$. Then there exists an R-bimodule isomorphism $M \otimes_R N \simeq N \otimes_R M$.*

Proof. Since $M|R$, by Lemma 2.11 we have the sequence of R-bimodule isomorphisms:

$$\begin{aligned} N \otimes_R M &\simeq N \otimes_R R \otimes_{\mathcal{Z}} C_M(R) \simeq N \otimes_{\mathcal{Z}} C_M(R) \simeq C_M(R) \otimes_{\mathcal{Z}} N \\ &\simeq C_M(R) \otimes_{\mathcal{Z}} N \simeq C_M(R) \otimes_{\mathcal{Z}} R \otimes_R N \simeq M \otimes_R N. \quad \square \end{aligned}$$

2.4. The group $\mathcal{P}(S/R)$

Let $R \subseteq S$ be an extension of rings with the same unity element. Following [27], let $\mathcal{M}(S/R)$ be the category with objects denoted by $P \overset{[\phi]}{\Longrightarrow} X$, where P is an R-bimodule, X is an S-bimodule and $\phi : P \longrightarrow X$ is an R-bilinear map such that the maps

$$\begin{array}{rccc} \bar{\phi}_r: & P \otimes_R S & \longrightarrow & X, \\ & p \otimes_R s & \longrightarrow & \phi(p)s \end{array} \quad \text{and} \quad \begin{array}{rccc} \bar{\phi}_l: & S \otimes_R P & \longrightarrow & X, \\ & s \otimes_R p & \longrightarrow & s\phi(p) \end{array} \tag{6}$$

are isomorphisms of R-S-bimodules and S-R-bimodules, respectively. A morphism from $P \xRightarrow{[\phi]} X$ to $Q \xRightarrow{[\psi]} Y$ in $\mathcal{M}(S/R)$ is a pair, where $\alpha : P \longrightarrow Q$ is R-bilinear, $\beta : X \longrightarrow Y$ is S-bilinear and the diagram

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\alpha"'] & X \arrow[d, "\beta"] \\ Q \arrow[r, "\psi"'] & Y \end{tikzcd}$$

is commutative. If α and β are isomorphisms of R-bimodules and S-bimodules, respectively, then (α, β) is an isomorphism between $P \xRightarrow{[\phi]} X$ and $Q \xRightarrow{[\psi]} Y$.

We denote by $\mathcal{P}(S/R)$ the set of the isomorphism classes of objects $[P] \xRightarrow{[\phi]} [X]$, where $[P] \in \mathbf{Pic}(R)$ and $[X] \in \mathbf{Pic}(S)$. By a slight abuse of notation $P \xRightarrow{[\phi]} X$ will also denote the isomorphism class of an object $P \xRightarrow{[\phi]} X$ of $\mathcal{M}(S/R)$.

Remark 2.13. By [44, Lemma 3.1], if $[P] \in \mathbf{Pic}(R)$ and $[X] \in \mathbf{Pic}(S)$, then $[P] \xRightarrow{[\phi]} [X] \in \mathcal{P}(S/R)$, provided that $\bar{\phi}_l$ (or $\bar{\phi}_r$) is an isomorphism.

By [45, Theorem 1.3], $\mathcal{P}(S/R)$ is a group, in which the product of the isomorphism class of $P \xRightarrow{[\phi]} X \in \mathcal{M}(S/R)$ by that of $Q \xRightarrow{[\psi]} Y \in \mathcal{M}(S/R)$ is given by the isomorphism class of

$$P \otimes_R Q \xRightarrow{[\phi\otimes\psi]} X \otimes_S Y \in \mathcal{M}(S/R),$$

where $\phi \otimes \psi : P \otimes_R Q \longrightarrow X \otimes_S Y$ is defined by $(\phi \otimes \psi)(p \otimes q) = \phi(p) \otimes \psi(q)$. Moreover, the inverse of the class of $P \xRightarrow{[\phi]} X$ is that of $[P^*] \xRightarrow{[\phi^*]} [X^*]$, where $\phi^*(f)(s\phi(p)) = sf(p)$, with $s \in S$, $p \in P$.

If R is a K-algebra, we define $\mathcal{P}_K(S/R) = \{ [P] \xRightarrow{[\phi]} [X] \in \mathcal{P}(S/R); [P] \in \mathbf{Pic}_K(R)\}$. It is easy to see that $\mathcal{P}_K(S/R)$ is a subgroup of $\mathcal{P}(S/R)$. Indeed, it is enough to check that if P is a central K-bimodule, then so is P^*. Let $k \in K$ and $f \in P^*$, then

$$(f \cdot k)(p) = f(p)k = kf(p) = f(kp) = f(pk) = (k \cdot f)(p),$$

for all $p \in P$. Therefore, $[P^*] \in \mathbf{Pic}_K(R)$, as desired.

3. Partial generalized crossed products

3.1. Partial actions, partial representations and partial cohomology

We shall need a concept of a partial group action on a semigroup which we give below and which is more general than that considered so far (see, in particular, [20]).

Definition 3.1. Let G be a group and S a semigroup. A partial action α of G on S is a family of subsemigroups S_x, $(x \in G)$, and semigroup isomorphisms $\alpha_x : S_{x^{-1}} \longrightarrow S_x$ which satisfy the following conditions:

(i) $S_1 = S$ and $\alpha_1 = Id_S$,
(ii) $\alpha_y^{-1}(S_y \cap S_{x^{-1}}) \subseteq S_{(xy)^{-1}}$,

(iii) $\alpha_x \circ \alpha_y(s) = \alpha_{xy}(s)$, for each $s \in \alpha_y^{-1}(S_y \cap S_{x^{-1}})$.

We shall write for simplicity $\alpha = (S_x, \alpha_x)$. As it can be seen in [15], conditions (ii) and (iii) of the above definition imply that $\alpha_x^{-1} = \alpha_{x^{-1}}$ and

$$\alpha_x(S_{x^{-1}} \cap S_y) = S_x \cap S_{xy}, \quad \text{for all } x, y \in G. \tag{7}$$

We say that the partial action $\alpha = (S_x, \alpha_x)$ is *unital* if each S_x is an ideal in S generated by an idempotent which is central in S, that is, $S_x = S1_x$, $1_x \in \mathcal{Z}(S)$, for all $x \in G$. In this case, $S_x \cap S_y = S1_x1_y$ and (7) implies that $\alpha_x(1_y1_{x^{-1}}) = 1_x1_{xy}$, for all $x, y \in G$. As a consequence,

$$\alpha_{xy}(s1_{y^{-1}x^{-1}})1_x = \alpha_x(\alpha_y(s1_{y^{-1}})1_{x^{-1}}), \quad \text{for all } \ x, y \in G \text{ and } s \in S.$$

The subsemigroup of the invariants of S with respect to the unital partial action α is defined by

$$S^\alpha = \{s \in S;\ \alpha_x(s1_{x^{-1}}) = s1_x, \text{ for all } x \in G\}.$$

We recall the next:

Definition 3.2. A partial representation of G in a monoid S is a map

$$\begin{array}{rccc} \theta: & G & \longrightarrow & S \\ & x & \longmapsto & \theta_x \end{array}$$

which satisfies the following properties:

(i) $\theta_{1_G} = 1_S$,
(ii) $\theta_x\theta_y\theta_{y^{-1}} = \theta_{xy}\theta_{y^{-1}}$, for all $x, y \in G$.
(iii) $\theta_{x^{-1}}\theta_x\theta_y = \theta_{x^{-1}}\theta_{xy}$, for all $x, y \in G$.

It follows from Definition 3.2 that

$$\theta_x\theta_{x^{-1}}\theta_x = \theta_x, \quad \text{for all } \ x \in G. \tag{8}$$

By [17] we know that the ε_x are idempotents such that

$$\varepsilon_x\varepsilon_y = \varepsilon_y\varepsilon_x \ \text{ and } \ \theta_x\varepsilon_y = \varepsilon_{xy}\theta_x \ \text{ for all } \ x, y \in G. \tag{9}$$

In particular,

$$\varepsilon_x\theta_x = \theta_x \quad \text{e} \quad \theta_x\varepsilon_{x^{-1}} = \theta_x, \text{ for all } x \in G. \tag{10}$$

Moreover, by (8)

$$\theta_x\theta_y = \theta_x\theta_{x^{-1}}\theta_x\theta_y = \theta_x\theta_{x^{-1}}\theta_{xy} = \varepsilon_x\theta_{xy}, \quad \text{for all } \ x, y \in G. \tag{11}$$

Applying (9) to (11), we obtain

$$\theta_x\theta_y = \varepsilon_x\theta_{xy} = \theta_{xy}\varepsilon_{y^{-1}}, \quad \text{for all } \ x, y \in G. \tag{12}$$

The need of a more general concept of a partial group action on a semigroup is justified by the following fact, which will be useful for us.

Proposition 3.3. *Let S be a monoid and let $\theta : G \longrightarrow S$ be a partial representation. Write $S_x = \varepsilon_x S \varepsilon_x$, $(x \in G)$. Then, $\alpha^* = (S_x, \alpha^*_x)$, where*

$$\begin{array}{rccc} \alpha^*_x : & S_{x^{-1}} & \longrightarrow & S_x \\ & s & \longmapsto & \theta_x s \theta_{x^{-1}}, \end{array}$$

is a partial action of G on S.

Proof. Clearly, $S_1 = S$ and $\alpha^*_1 = Id_S$. Observe that $s \in S_x$ if and only if $\varepsilon_x s \varepsilon_x = s$. Indeed, obviously, if $s = \varepsilon_x s \varepsilon_x$, then $s \in S_x$. On the other hand, if $s \in S_x$, then $s = \varepsilon_x s' \varepsilon_x$, for some $s' \in S$, and $\varepsilon_x s \varepsilon_x = \varepsilon_x \varepsilon_x s' \varepsilon_x \varepsilon_x = \varepsilon_x s' \varepsilon_x = s$, as desired.

By (10), we have

$$\alpha^*_x(s) = \theta_x s \theta_{x^{-1}} = \varepsilon_x \theta_x s \theta_{x^{-1}} \varepsilon_x = \varepsilon_x \alpha^*_x(s) \varepsilon_x,$$

and thus, $\alpha^*_x(s) \in S_x$. For $s, s' \in S_{x^{-1}}$, we see that

$$\alpha^*_x(ss') = \theta_x s s' \theta_{x^{-1}} = \theta_x s \varepsilon_{x^{-1}} s' \theta_{x^{-1}} = \theta_x s \theta_{x^{-1}} \theta_x s' \theta_{x^{-1}} = \alpha^*_x(s) \alpha^*_x(s').$$

Hence, α^*_x is a semigroup homomorphism. Given $s \in S_{x^{-1}}$, we obtain

$$(\alpha^*_{x^{-1}} \circ \alpha^*_x)(s) = \alpha^*_{x^{-1}}(\theta_x s \theta_{x^{-1}}) = \theta_{x^{-1}} \theta_x s \theta_{x^{-1}} \theta_x = \varepsilon_{x^{-1}} s \varepsilon_{x^{-1}} = s.$$

Analogously, $(\alpha^*_x \circ \alpha^*_{x^{-1}})(s) = s$, for all $s \in S_x$. Therefore, α^*_x is an isomorphism, whose inverse is $\alpha^*_{x^{-1}}$.

If $s \in S_y \cap S_{x^{-1}}$, then

$$\alpha^*_{y^{-1}}(s) = \theta_{y^{-1}} s \theta_y \overset{s \in S_{x^{-1}}}{=} \theta_{y^{-1}} \varepsilon_{x^{-1}} s \varepsilon_{x^{-1}} \theta_y \overset{(9)}{=} \varepsilon_{(xy)^{-1}} \theta_{y^{-1}} s \theta_y \varepsilon_{(xy)^{-1}} \in S_{(xy)^{-1}}.$$

Hence, $\alpha^*_{y^{-1}}(S_y \cap S_{x^{-1}}) \subseteq S_{(xy)^{-1}}$. If $s \in \alpha^*_{y^{-1}}(S_y \cap S_{x^{-1}}) \subseteq S_{(xy)^{-1}} \cap S_{y^{-1}}$, then using (11) and (12) we compute

$$\begin{aligned} (\alpha^*_x \circ \alpha^*_y)(s) &= \alpha^*_x(\theta_y s \theta_{y^{-1}}) = \theta_x \theta_y s \theta_{y^{-1}} \theta_{x^{-1}} = \theta_{xy} \varepsilon_{y^{-1}} s \varepsilon_{y^{-1}} \theta_{y^{-1}x^{-1}} \\ &= \theta_{xy} s \theta_{y^{-1}x^{-1}} = \alpha^*_{xy}(s). \end{aligned}$$

Consequently, α^* is a partial action. $\square$

Let K be a commutative ring (or a commutative monoid) and $\alpha = (D_x, \alpha_x)$ a unital partial action of G on K, where D_x is generated by the central idempotent 1_x. An *n-cochain*, with $n \in \mathbb{N}$, of G with values in K is a function $f : G^n \longrightarrow K$ such that $f(x_1, ..., x_n) \in \mathcal{U}(K 1_x 1_{x_1 x_2} ... 1_{x_1 x_2 ... x_n})$. For $n = 0$, we define a 0-cochain as an element in $\mathcal{U}(K)$.

Let $C^n(G, \alpha, K)$ be the set of all n-cochains of G with values in K. Then, $C^n(G, \alpha, K)$ is an abelian group with the multiplication defined point-wise, that is,

$$fg(x_1, ..., x_n) = f(x_1, ..., x_n) g(x_1, ..., x_n), \quad \text{with } x_1, ..., x_n \in G.$$

Clearly, $I(x_1, ..., x_n) = 1_{x_1} 1_{x_1 x_2} ... 1_{x_1 x_2 ... x_n}$ is the unit for this multiplication and

$$f^{-1}(x_1, ..., x_n) = f(x_1, ..., x_n)^{-1} \in \mathcal{U}(K 1_x 1_{x_1 x_2} ... 1_{x_1 x_2 ... x_n}), \text{ for } x_1, ..., x_n \in G.$$

Proposition 3.4. *[20, Proposition 1.5] The map $\delta^n : C^n(G,\alpha,K) \longrightarrow C^{n+1}(G,\alpha,K)$ defined by*

$$(\delta^n f)(x_1,...,x_{n+1}) = \alpha_{x_1}(f(x_2,...,x_{n+1})1_{x_1^{-1}}) \prod_{i=1}^{n} f(x_1,...,x_i x_{i+1},...,x_{n+1})^{(-1)^i} f(x_1,...,x_n)^{(-1)^{n+1}}, \tag{13}$$

where the inverse elements are taken in the corresponding ideals, is a group morphism such that

$$(\delta^{n+1} \circ \delta^n)(f)(x_1,...,x_{n+2}) = 1_{x_1}1_{x_1x_2}...1_{x_1x_2...x_{n+2}},$$

for all $f \in C^n(G,\alpha,K)$ and $x_1,...,x_{n+1} \in G$.

Remark 3.5. If $n = 0$, then $\delta^0 : \mathcal{U}(K) \longmapsto C^1(G,\alpha,K)$ is defined by

$$(\delta^0 k)(x) = \alpha_x(k1_{x^{-1}})k^{-1}, \quad \text{for each } k \in \mathcal{U}(K).$$

We define the groups

$$Z^n(G,\alpha,K) = \ker(\delta^n) \quad \text{and} \quad B^n(G,\alpha,K) = \operatorname{Im}(\delta^{n-1})$$

that are *the group of the partial n-cocycles and that of the partial n-coboundaries*, respectively. By Proposition 3.4, we have $B^n(G,\alpha,K) \subseteq Z^n(G,\alpha,K)$. Thus, we define the group of the partial n-cohomologies by

$$H^n(G,\alpha,K) = \frac{Z^n(G,\alpha,K)}{B^n(G,\alpha,K)}.$$

For $n = 0$ we set $H^0(G,\alpha,K) = Z^0(G,\alpha,K) = ker(\delta^0)$.

Let $f, f' \in Z^n(G,\alpha,K)$. We say that f and f' are *cohomologous* if there is $g \in C^{n-1}(G,\alpha,K)$ such that $f = f'(\delta^n g)$. In this case, $[f] = [f']$ in $H^n(G,\alpha,K)$.

An n-cocycle f is called *normalized* if

$$f(1,x_1,...,x_{n-1}) = f(x_1,1,...,x_{n-1}) = \cdots = f(x_1,...,x_{n-1},1) = 1_{x_1}1_{x_1x_2}...1_{x_1...x_{n-1}},$$

for all $x_1, x_2, \ldots, x_{n-1} \in G$.

Remark 3.6. Any partial 1-cocycle σ is normalized. Indeed, we have $\alpha_x(\sigma_y 1_{x^{-1}})\sigma_{xy}^{-1}\sigma_x = 1_x 1_{xy}$, for all $x, y \in G$. In particular, taking $x = y = 1$, we obtain $\sigma_1^2 = \sigma_1$, which implies that $\sigma_1 = 1$. Therefore, σ is normalized. By [20, Remark 2.6] if $\sigma \in Z^2(G,\alpha,K)$, then there is a normalized $\widetilde{\sigma} \in Z^2(G,\alpha,K)$ such that $\sigma = \widetilde{\sigma}(\delta^1\beta)$, for some $\beta \in C^1(G,\alpha,K)$. Therefore, σ is cohomologous to a normalized partial 2-cocycle.

Observe also that by [20, Remark 2.6], if $\sigma \in Z^2(G,\alpha,\mathcal{Z})$, then there exists a normalized $\widetilde{\sigma} \in Z^2(G,\alpha,\mathcal{Z})$ such that $\sigma = \widetilde{\sigma}(\delta^1\beta)$, for some $\beta \in C^1(G,\alpha,\mathcal{Z})$. Hence, σ is cohomologous to a normalized partial 2-cocycle.

*3.2. Unital partial representations $G \to$ **PicS**(R)*

Consider a partial representation

$$\Theta : G \longrightarrow \mathbf{PicS}(R)$$

of a group G into the monoid $\mathbf{PicS}(R)$. For each $x \in G$ fix a representative Θ_x of the isomorphism class $\Theta(x) \in \mathbf{PicS}(R)$, i.e. $\Theta(x) = [\Theta_x]$. We say that Θ is *unital* if $[\varepsilon_x] = [\Theta_x][\Theta_{x^{-1}}] = [R1_x]$, where 1_x is a central idempotent of R, for all $x \in G$. Thus, there is an R-bimodule isomorphism $\Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$. In the following auxiliary fact we gather some isomorphisms involving Θ_x for further use.

Lemma 3.7. *Let $\Theta : G \longrightarrow \mathbf{PicS}(R)$ be a unital partial representation with $\Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, $(x \in G)$. Then, for all $x, y \in G$, we have the following R-bimodule isomorphisms:*

(i) $\Theta_x \otimes_R R1_y \simeq R1_{xy} \otimes_R \Theta_x$. *In particular,*

$$R1_x \otimes_R \Theta_x \simeq \Theta_x \text{ and } \Theta_x \otimes_R R1_{x^{-1}} \simeq \Theta_x, \text{ for all } x \in G. \tag{14}$$

(ii) $\Theta_x \otimes_R \Theta_y \simeq R1_x \otimes_R \Theta_{xy}$ *and* $\Theta_x \otimes_R \Theta_y \simeq \Theta_{xy} \otimes_R R1_{y^{-1}}$,
(iii) $\Theta_x \otimes_R \Theta_y \otimes_R \Theta_{(xy)^{-1}} \simeq R1_x 1_{xy}$,
(iv) $\Theta_x \otimes_R \Theta_y \simeq \Theta_x \otimes_R \Theta_y \otimes_R R1_{y^{-1}x^{-1}}$.

Proof. Item (i) is a direct consequence of (9), whereas (ii) follows from (11) and (12). For (iii) we have

$$\Theta_x \otimes_R \Theta_y \otimes_R \Theta_{(xy)^{-1}} \overset{(ii)}{\simeq} R1_x \otimes_R \Theta_{xy} \otimes_R \Theta_{(xy)^{-1}} \simeq R1_x \otimes_R R1_{xy} \simeq R1_x 1_{xy}.$$

As to (iv):

$$\begin{aligned}\Theta_x \otimes_R \Theta_y \otimes_R R1_{y^{-1}x^{-1}} &\simeq \Theta_x \otimes_R \Theta_y \otimes_R \Theta_{(xy)^{-1}} \otimes_R \Theta_{xy} \simeq \Theta_x \otimes_R \Theta_y \otimes_R \Theta_{y^{-1}} \otimes_R \Theta_{x^{-1}} \otimes_R \Theta_{xy} \\ &\simeq \Theta_x \otimes_R \Theta_y \otimes_R \Theta_{y^{-1}} \otimes_R \Theta_{x^{-1}} \otimes_R \Theta_x \otimes_R \Theta_y \simeq \Theta_x \otimes_R R1_y \otimes_R R1_{x^{-1}} \otimes_R \Theta_y \\ &\simeq \Theta_x \otimes_R R1_{x^{-1}} \otimes_R R1_y \otimes_R \Theta_y \simeq \Theta_x \otimes_R \Theta_y. \quad \square\end{aligned}$$

Using the isomorphism in (4), and (i) and (ii) of Lemma 3.7, we have:

$$1_{xy}\Theta_x \simeq \Theta_x 1_y \quad \text{and} \quad \Theta_x \otimes_R \Theta_y \simeq 1_x \Theta_{xy}, \quad \text{for all } x, y \in G, \tag{15}$$

as R-bimodules. In particular,

$$1_x \Theta_x \simeq \Theta_x \quad \text{and} \quad \Theta_x 1_{x^{-1}} \simeq \Theta_x, \tag{16}$$

Lemma 3.8. *Let $\Theta : G \longrightarrow \mathbf{PicS}(R)$ be a unital partial representation with $\varepsilon_x = \Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, where $\Theta(x) = [\Theta_x]$ for all $x \in G$. Then,*

$$u_x 1_y = 1_{xy} u_x 1_y = 1_{xy} u_x, \quad \text{for all } x, y \in G \text{ and } u_x \in \Theta_x. \tag{17}$$

In particular, $1_{xy}\Theta_x = \Theta_x 1_y$, for all $x, y \in G$.

Proof. By (15) there exists an R-bimodule isomorphism $\kappa_{x,y} : \Theta_x 1_y \longrightarrow 1_{xy}\Theta_x$. Given $u_x \in \Theta_x$, there is $v_x \in \Theta_x$ such that $\kappa_{x,y}(u_x 1_y) = 1_{xy} v_x$. Then

$$\kappa_{x,y}(1_{xy} u_x 1_y) = 1_{xy} v_x = \kappa_{x,y}(u_x 1_y).$$

Since $\kappa_{x,y}$ is an isomorphism, then $1_{xy} u_x 1_y = u_x 1_y$. On the other hand, there exists $v'_x \in \Theta_x$ such that $\kappa^{-1}_{x,y}(1_{xy} u_x) = v'_x 1_y$ and

$$\kappa_{x,y}^{-1}(1_{xy}u_x1_y) = v'_x1_y = \kappa_{x,y}^{-1}(1_{xy}u_x).$$

Thus, $1_{xy}u_x1_y = 1_{xy}u_x$. Therefore,

$$u_x1_y = 1_{xy}u_x1_y = 1_{xy}u_x, \text{ for all } u_x \in \Theta_x. \quad \square$$

It follows from Lemma 3.8 that

$$1_xu_x = u_x \text{ and } u_x1_{x^{-1}} = u_x, \text{ for all } x \in G, u_x \in \Theta_x. \tag{18}$$

The maps $R1_x \otimes_R \Theta_x \simeq \Theta_x$ and $\Theta_x \otimes_R R1_{x^{-1}} \simeq \Theta_x$ defined via the right R-action and the left R-action on Θ_x, respectively, are isomorphisms. Thus, Θ_x is a unital $(R1_x, R1_{x^{-1}})$-bimodule and there exist R-bimodule isomorphisms

$$\tau_x : \Theta_x \otimes_R \Theta_{x^{-1}} \longrightarrow R1_x \text{ and } \tau_{x^{-1}} : \Theta_{x^{-1}} \otimes_R \Theta_x \longrightarrow R1_{x^{-1}}. \tag{19}$$

Note that $\Theta_x \otimes_R \Theta_{x^{-1}} = \Theta_x \otimes_{R1_{x^{-1}}} \Theta_{x^{-1}}$ as $(R1_x, R1_{x^{-1}})$-bimodules. In fact, by (18)

$$u_xr \otimes_{R1_{x^{-1}}} u_{x^{-1}} = u_x1_{x^{-1}}r \otimes_{R1_{x^{-1}}} u_{x^{-1}} = u_x \otimes_{R1_{x^{-1}}} r1_{x^{-1}}u_{x^{-1}} = u_x \otimes_{R1_{x^{-1}}} ru_{x^{-1}},$$

for all $u_x \in \Theta_x, u_{x^{-1}} \in \Theta_{x^{-1}}$ and $r \in R$. Thus, we can choose the isomorphisms τ_x and $\tau_{x^{-1}}$ such that $(R1_x, R1_{x^{-1}}, \Theta_x, \Theta_{x^{-1}}, \tau_x, \tau_{x^{-1}})$ is a Morita context, that is, τ_x and $\tau_{x^{-1}}$ satisfy:

$$\tau_x(u_x \otimes u_{x^{-1}})u_x = u_x\tau_{x^{-1}}(u_{x^{-1}} \otimes u'_x) \text{ and } \tau_{x^{-1}}(u_{x^{-1}} \otimes u_x)u'_{x^{-1}} = u_{x^{-1}}\tau_x(u_x \otimes u'_{x^{-1}}), \tag{20}$$

for all $u_x, u'_x \in \Theta_x$. For simplicity, we write $\tau_x(u_x \otimes u_{x^{-1}}) = u_xu_{x^{-1}}$, where $u_x \in \Theta_x$ and $u_{x^{-1}} \in \Theta_{x^{-1}}$. Thus, by (20) we have the associativity:

$$u_xu_{x^{-1}}u'_x = (u_xu_{x^{-1}})u'_x = u_x(u_{x^{-1}}u'_x) \tag{21}$$

for $u_x, u'_x \in \Theta_x$ and $u_{x^{-1}} \in \Theta_{x^{-1}}$.

Lemma 3.9. *For all $u_x \in \Theta_x$, with $x \in G$, and $r \in \mathcal{Z}$, the following equality holds:*

$$(u_x(u_yu_{y^{-1}})u_{x^{-1}})u_{xy}ru_{(xy)^{-1}} = (u_x(u_yru_{y^{-1}})u_{x^{-1}})u_{xy}u_{(xy)^{-1}}. \tag{22}$$

Proof. Firstly, by (15) we have

$$\Theta_x \otimes \Theta_y \otimes \Theta_{(xy)^{-1}} \simeq R1_x \otimes \Theta_{xy} \otimes \Theta_{(xy)^{-1}} \simeq R1_x \otimes R1_{xy} \simeq R1_x1_{xy}.$$

Let φ be the composition of the above R-bimodule isomorphisms. For each $r \in \mathcal{Z}$ we have $r\varphi(u_x \otimes u_y \otimes u_{(xy)^{-1}}) = \varphi(u_x \otimes u_y \otimes u_{(xy)^{-1}})r$. Since φ is R-bilinear, it follows that

$$\varphi(ru_x \otimes u_y \otimes u_{(xy)^{-1}}) = \varphi(u_x \otimes u_y \otimes u_{(xy)^{-1}}r).$$

Moreover, since φ is an isomorphism, we have

$$ru_x \otimes u_y \otimes u_{(xy)^{-1}} = u_x \otimes u_y \otimes u_{(xy)^{-1}}r \tag{23}$$

for all $r \in \mathcal{Z}$. Now, note that

$$(u_x(u_yu_{y^{-1}})u_{x^{-1}})u_{xy}ru_{(xy)^{-1}} = \tau_{xy}(\tau_x(u_x\tau_y(u_y \otimes u_{y^{-1}}) \otimes u_{x^{-1}})u_{xy} \otimes u_{(xy)^{-1}}).$$

Rewriting the right-hand side of the above equality and using (23), we have

$$\begin{aligned}(u_x(u_yu_{y^{-1}})u_{x^{-1}})u_{xy}ru_{(xy)^{-1}} &= \tau_{xy}(\tau_x(u_x\tau_y(u_y \otimes u_{y^{-1}}) \otimes u_{x^{-1}})u_{xy}r \otimes u_{(xy)^{-1}})\\ &= [\tau_{xy} \circ (\tau_x \otimes \Theta_{xy} \otimes \Theta_{(xy)^{-1}}) \circ (\Theta_x \otimes \tau_y \otimes \Theta_{x^{-1}} \otimes \Theta_{xy} \otimes \Theta_{(xy)^{-1}})]\\ &\quad (u_x \otimes u_y \otimes u_{y^{-1}} \otimes u_{x^{-1}} \otimes u_{xy}r \otimes u_{(xy)^{-1}})\\ &= [\tau_{xy} \circ (\tau_x \otimes \Theta_{xy} \otimes \Theta_{(xy)^{-1}}) \circ (\Theta_x \otimes \tau_y \otimes \Theta_{x^{-1}} \otimes \Theta_{xy} \otimes \Theta_{(xy)^{-1}})]\\ &\quad (u_x \otimes u_yr \otimes u_{y^{-1}} \otimes u_{x^{-1}} \otimes u_{xy} \otimes u_{(xy)^{-1}})\\ &= \tau_{xy}(\tau_x(u_x\tau_y(u_yr \otimes u_{y^{-1}}) \otimes u_{x^{-1}})u_{xy} \otimes u_{(xy)^{-1}})\\ &= (u_x(u_yru_{y^{-1}})u_{x^{-1}})u_{xy}u_{(xy)^{-1}}. \quad \square\end{aligned}$$

For the rest of this section, we will fix a unital partial representation $\Theta : G \longrightarrow \mathbf{PicS}(R)$ with $\Theta(x) = [\Theta_x]$ and $\varepsilon_x = \Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$. Denote by $\sum\limits_{(x)} \omega_x \otimes \omega_{x^{-1}} \in \Theta_x \otimes_R \Theta_{x^{-1}}$ the inverse image of 1_x under the isomorphism τ_x, that is, $\sum\limits_{(x)} \omega_x\omega_{x^{-1}} = 1_x$.

Lemma 3.10. *There is a group homomorphism*

$$\begin{array}{ccc}\mathbf{Aut}_{R-R}(1_y\Theta_x) & \xrightarrow{\widetilde{(-)}} & \mathcal{U}(\mathcal{Z}1_x1_y)\\ \sigma & \longmapsto & \widetilde{\sigma},\end{array}$$

where

$$\widetilde{\sigma} = \sum_{(x)} \sigma(1_y\omega_x)\omega_{x^{-1}}. \tag{24}$$

Moreover,

$$\sigma(1_yu_x) = \widetilde{\sigma}u_x, \quad \textit{for all } u_x \in \Theta_x. \tag{25}$$

Proof. Firstly, since σ is R-bilinear, we have

$$\sigma(1_yu_x)u_{x^{-1}}v_xv_{x^{-1}} = \sigma(1_yu_xu_{x^{-1}}v_x)v_{x^{-1}}, \tag{26}$$

for all $u_x, v_x \in \Theta_x$. Let us check first that $\widetilde{\sigma}$ does not depend on the choice of the decomposition of 1_x. Let $\sum\limits_{(\widetilde{x})} \widetilde{\omega}_x\widetilde{\omega}_{x^{-1}} = 1_x$, with $\widetilde{\omega}_x \in \Theta_x$ and $\widetilde{\omega}_{x^{-1}} \in \Theta_{x^{-1}}$, be another decomposition of 1_x. Then by (26)

$$\begin{aligned}\widetilde{\sigma} &= \sum_{(x)} \sigma(1_y\omega_x)\omega_{x^{-1}} = \sum_{(x)} \sigma(1_y\omega_x)\omega_{x^{-1}}1_x = \sum_{(x),(\widetilde{x})} \sigma(1_y\omega_x)\omega_{x^{-1}}\omega_x\widetilde{\omega}_{x^{-1}}\\ &= \sum_{(x),(\widetilde{x})} \sigma(1_y\omega_x\omega_{x^{-1}}\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}} = \sum_{(\widetilde{x})} \sigma(1_y1_x\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}} = \sum_{(\widetilde{x})} \sigma(1_y\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}}.\end{aligned}$$

Clearly $\widetilde{\sigma} \in R1_x1_y$. Now, for $r \in R$:, using the previous argument, we have

$$\widetilde{\sigma}r=\sum_{(x)}\sigma(1_y\omega_x)\omega_{x^{-1}}r=\sum_{(x)}\sigma(1_y\omega_x)\omega_{x^{-1}}r1_x=\sum_{(\widetilde{x})}\sigma(1_y1_xr\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}}=\sum_{(\widetilde{x})}r\sigma(1_y\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}}=r\widetilde{\sigma}.$$

Thus, $\widetilde{\sigma}\in\mathcal{Z}1_x1_y$. It remains to verify that $\widetilde{\sigma}$ is an invertible element of $\mathcal{Z}1_x1_y$. Given $\sigma^{-1}\in\mathbf{Aut}_{R-R}(1_y\Theta_x)$, take $\widetilde{\sigma^{-1}}\in\mathcal{Z}1_x1_y$. Then

$$\begin{aligned}\widetilde{\sigma}\widetilde{\sigma^{-1}}&=\sum_{(x),(\widetilde{x})}\sigma(1_y\omega_x)\omega_{x^{-1}}\sigma^{-1}(1_y\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}}=\sum_{(x),(\widetilde{x})}\sigma(1_y\omega_x\omega_{x^{-1}}\sigma^{-1}(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}\\&=\sum_{(\widetilde{x})}\sigma(1_y1_x\sigma^{-1}(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}=\sum_{(\widetilde{x})}\sigma(\sigma^{-1}(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}=\sum_{(\widetilde{x})}1_y\widetilde{\omega}_x\widetilde{\omega}_{x^{-1}}=1_y1_x.\end{aligned}$$

Analogously, $\widetilde{\sigma^{-1}}\widetilde{\sigma}=1_x1_y$. Therefore, $\widetilde{\sigma}\in\mathcal{U}(\mathcal{Z}1_x1_y)$.

Moreover, if $u_x\in\Theta_x$, then:

$$\widetilde{\sigma}u_x=\sum_{(x)}\sigma(1_y\omega_x)\omega_{x^{-1}}u_x=\sum_{(x)}\sigma(1_y\omega_x\omega_{x^{-1}}u_x)=\sigma(1_y1_xu_x)=\sigma(1_yu_x),$$

and (25) follows.

Given $\sigma,\gamma\in\mathbf{Aut}_{R-R}(1_y\Theta_x)$, we have

$$\begin{aligned}\widetilde{\sigma}\widetilde{\gamma}&=\sum_{(x),(\widetilde{x})}\sigma(1_y\omega_x)\omega_{x^{-1}}\gamma(1_y\widetilde{\omega}_x)\widetilde{\omega}_{x^{-1}}=\sum_{(x),(\widetilde{x})}\sigma(1_y\omega_x\omega_{x^{-1}}\gamma(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}\\&=\sum_{(\widetilde{x})}\sigma(1_y1_x\gamma(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}=\sum_{(\widetilde{x})}\sigma(\gamma(1_y\widetilde{\omega}_x))\widetilde{\omega}_{x^{-1}}=\widetilde{\sigma\circ\gamma}.\end{aligned}$$

Therefore, $\widetilde{(-)}$ is a group morphism.

Now, given $r\in\mathcal{U}(\mathcal{Z}1_x1_y)$, consider the map

$$\begin{array}{rccl}\sigma_r: & 1_y\Theta_x & \longrightarrow & 1_y\Theta_x,\\ & v_x & \longmapsto & rv_x.\end{array}$$

Clearly σ_r is right R-linear. Since $r\in\mathcal{Z}$ then σ_r is left R-linear. Indeed, if $r'\in R$, then

$$\sigma_r(r'v_x)=rr'v_x=r'rv_x=r'\sigma_r(v_x).$$

Let r^{-1} be the inverse element of r in $\mathcal{Z}1_x1_y$ and let $\sigma_{r^{-1}}:1_y\Theta_x\to1_y\Theta_x$ be defined by $\sigma_{r^{-1}}(u_x)=r^{-1}u_x$. Given $v_x\in1_y\Theta_x$, we have that

$$\sigma_{r^{-1}}(\sigma_r(v_x))=\sigma_{r^{-1}}(rv_x)=r^{-1}(rv_x)=(r^{-1}r)v_x=1_x1_yv_x=v_x.$$

Analogously, $\sigma_r(\sigma_{r^{-1}}(v_x))=v_x$, for all $v_x\in1_y\Theta_x$. Thus, σ_r is a bijection with $\sigma_r^{-1}=\sigma_{r^{-1}}$. It follows that $\sigma_r\in\mathbf{Aut}_{R-R}(1_y\Theta_x)$. Notice now that

$$\widetilde{\sigma_r}=\sum_{(x)}\sigma_r(1_y\omega_x)\omega_{x^{-1}}=\sum_{(x)}r1_y\omega_x\omega_{x^{-1}}=r1_y1_x=r.$$

Therefore, $\widetilde{(-)}$ is onto. Let $\gamma,\sigma\in\mathbf{Aut}_{R-R}(1_y\Theta_x)$ such that $\widetilde{\gamma}=\widetilde{\sigma}$. For any $v_x\in1_y\Theta_x$, by (25) it follows that

$$\gamma(v_x)=\widetilde{\gamma}v_x=\widetilde{\sigma}v_x=\sigma(v_x).$$

Thus, $\gamma = \sigma$ and, consequently $\widetilde{(-)}$ is injective and, therefore, it is a group isomorphism. $\square$

Proposition 3.11. *Let $D_x = \mathcal{Z}1_x$ be the ideal of $\mathcal{Z}$ generated by the central idempotent 1_x and let $\alpha_x : D_{x^{-1}} \mapsto D_x$ be defined by*

$$\alpha_x(r) = \sum_{(x)} \omega_x r \omega_{x^{-1}}.$$

Then, $\alpha = (D_x, \alpha_x)_{x\in G}$ is a partial action of G on $\mathcal{Z}$.

Proof. First, let us check that α_x does not depend on the choice of the decomposition of 1_x. Let $\sum_{(\widetilde{x})} \widetilde{\omega}_x \widetilde{\omega}_{x^{-1}} = 1_x$, with $\widetilde{\omega}_x \in \Theta_x$ and $\widetilde{\omega}_{x^{-1}} \in \Theta_{x^{-1}}$, be another decomposition of 1_x. Given $r \in \mathcal{Z}$, using (18) and (21), we have

$$\begin{aligned}\alpha_x(r1_{x^{-1}}) &= \sum_{(x)} \omega_x r \omega_{x^{-1}} \overset{(18)}{=} \sum_{(x)} \omega_x r \omega_{x^{-1}} 1_x = \sum_{(x),(\widetilde{x})} \omega_x r \omega_{x^{-1}} \widetilde{\omega}_x \widetilde{\omega}_{x^{-1}} \\ &\overset{r\in\mathcal{Z}}{=} \sum_{(x),(\widetilde{x})} \omega_x \omega_{x^{-1}} \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}} = \sum_{(\widetilde{x})} 1_x \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}} \overset{(18)}{=} \sum_{(\widetilde{x})} \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}}.\end{aligned}$$

Now, for $r \in \mathcal{Z}$ and for any $r' \in R$, we see that

$$\begin{aligned}\alpha_x(r1_{x^{-1}})r' &= \sum_{(x)} \omega_x r \omega_{x^{-1}} r' = \sum_{(x)} \omega_x r \omega_{x^{-1}} 1_x r' = \sum_{(x),(\widetilde{x})} \omega_x r \omega_{x^{-1}} r' \widetilde{\omega}_x \widetilde{\omega}_{x^{-1}} \\ &\overset{r\in\mathcal{Z}}{=} \sum_{(x),(\widetilde{x})} \omega_x \omega_{x^{-1}} r' \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}} = \sum_{(\widetilde{x})} 1_x r' \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}} = \sum_{(\widetilde{x})} r' \widetilde{\omega}_x r \widetilde{\omega}_{x^{-1}} = r' \alpha_x(r1_{x^{-1}}).\end{aligned}$$

Thus, $\alpha_x(r1_{x^{-1}}) \in D_x = \mathcal{Z}1_x$ and α_x is well-defined. Let $\sum_{(\overline{x})} \overline{\omega}_{x^{-1}} \otimes \overline{\omega}_x \in \Theta_{x^{-1}} \otimes_R \Theta_x$ be the inverse image of $1_{x^{-1}}$ under the isomorphism $\tau_{x^{-1}}$, that is, $\sum_{(\overline{x})} \overline{\omega}_{x^{-1}} \overline{\omega}_x = 1_{x^{-1}}$. Given $r \in D_x$, we have

$$\begin{aligned}(\alpha_x \circ \alpha_{x^{-1}})(r) &= \alpha_x \left(\sum_{(\overline{x})} \overline{\omega}_{x^{-1}} r \overline{\omega}_x \right) = \sum_{(x),(\overline{x})} \omega_x \overline{\omega}_{x^{-1}} r \overline{\omega}_x \omega_{x^{-1}} \overset{r\in\mathcal{Z}}{=} \sum_{(x),(\overline{x})} r \omega_x \overline{\omega}_{x^{-1}} \overline{\omega}_x \omega_{x^{-1}} \\ &= \sum_{(x)} r \omega_x 1_{x^{-1}} \omega_{x^{-1}} = \sum_{(x)} r \omega_x \omega_{x^{-1}} = r1_x = r.\end{aligned}$$

Analogously, given $r \in D_{x^{-1}}$, we obtain $(\alpha_{x^{-1}} \circ \alpha_x)(r) = r$. Therefore, $\alpha_x^{-1} = \alpha_{x^{-1}}$.

For $r, s \in D_x$ we see that

$$\begin{aligned}\alpha_x(r)\alpha_x(s) &= \sum_{(x),(\widetilde{x})} \omega_x r \omega_{x^{-1}} \widetilde{\omega}_x s \widetilde{\omega}_{x^{-1}} \overset{s\in\mathcal{Z}}{=} \sum_{(x),(\widetilde{x})} \omega_x r s \omega_{x^{-1}} \widetilde{\omega}_x \widetilde{\omega}_{x^{-1}} \\ &= \sum_{(x)} \omega_x r s \omega_{x^{-1}} 1_x = \sum_{(x)} \omega_x r s \omega_{x^{-1}} = \alpha_x(rs).\end{aligned}$$

Thus, α_x is a ring isomorphism.

Write $1_{y^{-1}} = \sum_{(\overline{y})} \overline{\omega}_{y^{-1}} \overline{\omega}_y$. Given $r \in D_y \cap D_{x^{-1}}$, using (17), we obtain

$$\alpha_{y^{-1}}(r) = \sum_{(\overline{y})} \overline{\omega}_{y^{-1}} r \overline{\omega}_y = \sum_{(\overline{y})} \overline{\omega}_{y^{-1}} r 1_{x^{-1}} \overline{\omega}_y = \sum_{(\overline{y})} \overline{\omega}_{y^{-1}} r \overline{\omega}_y 1_{(xy)^{-1}} \in D_{(xy)^{-1}}.$$

Hence, $\alpha_{y^{-1}}(D_y \cap D_{x^{-1}}) \subseteq D_{(xy)^{-1}}$. Using again (17),

$$1_x 1_{xy} = \sum_{(x)} \omega_x \omega_{x^{-1}} 1_{xy} = \sum_{(x)} \omega_x 1_y \omega_{x^{-1}} = \sum_{(x),(y)} \omega_x(\omega_y \omega_{y^{-1}})\omega_{x^{-1}}. \tag{27}$$

Write $\sum_{(xy)} \omega_{xy}\omega_{(xy)^{-1}} = 1_{xy}$ and take $r \in D_{y^{-1}} \cap D_{(xy)^{-1}}$. By (22) and (27) we compute

$$\begin{aligned}
\alpha_{xy}(r) &= \sum_{(xy)} \omega_{xy} r \omega_{(xy)^{-1}} \overset{r\in D_{y^{-1}}}{=} \sum_{(xy)} 1_{xy}\omega_{xy} 1_{y^{-1}} r \omega_{(xy)^{-1}} \overset{(17)}{=} \sum_{(xy)} 1_{xy} 1_x \omega_{xy} r \omega_{(xy)^{-1}} \\
&\overset{(27)}{=} \sum_{\substack{(x),(y),\\(xy)}} (\omega_x(\omega_y\omega_{y^{-1}})\omega_{x^{-1}})\omega_{xy} r \omega_{(xy)^{-1}} = \sum_{\substack{(x),(y),\\(xy)}} (\omega_x(\omega_y r \omega_{y^{-1}})\omega_{x^{-1}})\omega_{xy}\omega_{(xy)^{-1}} \\
&= \sum_{(x),(y)} (\omega_x(\omega_y r \omega_{y^{-1}})\omega_{x^{-1}}) 1_{xy} \overset{(17)}{=} \sum_{(x),(y)} (\omega_x(\omega_y r \omega_{y^{-1}}) 1_y \omega_{x^{-1}}) \\
&= \sum_{(x),(y)} (\omega_x(\omega_y r \omega_{y^{-1}})\omega_{x^{-1}}) = \alpha_x \circ \alpha_y(r).
\end{aligned}$$

Therefore, $\alpha = (D_x, \alpha_x)_{x\in G}$ is a partial action of G on $\mathcal{Z}$. □

Lemma 3.12. *Let $\alpha = (D_x, \alpha_x)$ be the partial action of G on $\mathcal{Z}$ constructed in Proposition 3.11 and M an R-bimodule. Then:*

(i) *For all $u_x \in \Theta_x$ and $r \in \mathcal{Z}$ we have*

$$\alpha_x(r1_{x^{-1}})u_x = u_x r. \tag{28}$$

(ii) *If $M|\Theta_x$, then for all $m \in M$ and $r \in \mathcal{Z}$, we have*

$$\alpha_x(r1_{x^{-1}})m = mr \tag{29}$$

Proof. (i) Given $r \in \mathcal{Z}$ and $u_x \in \Theta_x$, by (18) and (21) we have

$$\alpha_x(r1_{x^{-1}})u_x = \sum_{(x)} (\omega_x r \omega_{x^{-1}})u_x = \sum_{(x)} \omega_x\omega_{x^{-1}} u_x r = 1_x u_x r = u_x r.$$

(ii) Let $f_i : M \to \Theta_x$ and $g_i : \Theta_x \to M$, $i = 1, 2, ..., n$, be R-bimodule morphisms such that $\sum_{i=1}^n g_i f_i = Id_M$. Given $m \in M$ and $r \in \mathcal{Z}$, using (28) we obtain

$$\begin{aligned}
mr &= \sum_{i=1}^n g_i(f_i(m))r = \sum_{i=1}^n g_i(\underbrace{f_i(m)}_{\in\Theta_x} r) = \sum_{i=1}^n g_i(\alpha_x(r1_{x^{-1}})f_i(m)) \\
&= \sum_{i=1}^n \alpha_x(r1_{x^{-1}})g_i(f_i(m)) = \alpha_x(r1_{x^{-1}})m. \quad \square
\end{aligned}$$

We denote by $H^n_\Theta(G, \alpha, \mathcal{Z})$ the group of partial n-cohomologies, where α is the partial action induced by the untial partial representation Θ.

We shall define now a partial action of G on $\mathbf{PicS}(R)$. Let $\mathcal{X}_x = [R1_x]\mathbf{PicS}(R)[R1_x]$ and

$$\begin{array}{rccl} \alpha^*_x : & \mathcal{X}_{x^{-1}} & \longrightarrow & \mathcal{X}_x, \\ & [P] & \longmapsto & [\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}]. \end{array}$$

It follows by Proposition 3.3 that $\alpha^* = (\mathcal{X}_x, \alpha^*_x)$ is a partial action of G on $\mathbf{PicS}(R)$. Observe that if $[P] \in \mathbf{PicS}_{\mathcal{Z}}(R)$, then by Corollary 2.12, we obtain that $P \otimes_R R1_x \simeq R1_x \otimes_R P$, as R-bimodules, for all $x \in G$. Hence, $[R1_x]$ is a central idempotent in $\mathbf{PicS}_{\mathcal{Z}}(R)$. Moreover, if $[P] \in \mathbf{PicS}_{\mathcal{Z}}(R)$, then $[\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}] \in \mathbf{PicS}_{\mathcal{Z}}(R)$. Indeed, it is enough to verify that $\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}$ is also a central $\mathcal{Z}$-bimodule. Let $p \in P$, $u_x \in \Theta_x, u_{x^{-1}} \in \Theta_{x^{-1}}$ and $r \in \mathcal{Z}$. Since P is central over $\mathcal{Z}$, using (28) we obtain:

$$\begin{aligned} u_x \otimes p \otimes u_{x^{-1}} r &= u_x \otimes p \otimes \alpha_{x^{-1}}(r1_x) u_{x^{-1}} = u_x \alpha_{x^{-1}}(r1_x) \otimes p \otimes u_{x^{-1}} \\ &= r1_x u_x \otimes p \otimes u_{x^{-1}} = r u_x \otimes p \otimes u_{x^{-1}}. \end{aligned}$$

Thus, restricting the above partial action to $\mathbf{PicS}_{\mathcal{Z}}(R)$, we have that $\mathcal{X}_x = \mathbf{PicS}_{\mathcal{Z}}(R)[R1_x]$, i.e. $\mathcal{X}_x$ is the ideal in $\mathbf{PicS}_{\mathcal{Z}}(R)$ generated by the central idempotent $[R1_x]$. Furthermore,

$$\begin{array}{rccl} \alpha^*_x : & \mathbf{PicS}_{\mathcal{Z}}(R)[R1_{x^{-1}}] & \longrightarrow & \mathbf{PicS}_{\mathcal{Z}}(R)[R1_x], \\ & [P][R1_{x^{-1}}] & \longmapsto & [\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}] \end{array}$$

is well-defined and, consequently, $\alpha^* = (\mathcal{X}_x, \alpha^*_x)$ is a partial action of G on $\mathbf{PicS}_{\mathcal{Z}}(R)$.

The subsemigroup of invariants of $\mathbf{PicS}_{\mathcal{Z}}(R)$ is given by

$$\mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} = \{[P] \in \mathbf{PicS}_{\mathcal{Z}}(R);\ \Theta_x \otimes_R P \simeq P \otimes_R \Theta_x,\ \text{for all } x \in G\}. \tag{30}$$

Indeed, $[P] \in \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$, if and only if $\alpha^*_x([P][R1_{x^{-1}}]) = [P][R1_x]$, for all $x \in G$. It follows that

$$\begin{aligned} [P] \in \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} &\Leftrightarrow \Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}} \simeq P \otimes_R R1_x, \quad \forall x \in G; \\ &\Leftrightarrow \Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}} \otimes_R \Theta_x \simeq P \otimes_R R1_x \otimes_R \Theta_x, \quad \forall x \in G; \\ &\Leftrightarrow \Theta_x \otimes_R P \otimes_R R1_{x^{-1}} \simeq P \otimes_R \Theta_x, \quad \forall x \in G; \\ &\Leftrightarrow \Theta_x \otimes_R R1_{x^{-1}} \otimes_R P \simeq P \otimes_R \Theta_x, \quad \forall x \in G; \\ &\Leftrightarrow \Theta_x \otimes_R P \simeq P \otimes_R \Theta_x, \quad x \in G. \end{aligned}$$

3.3. Partial generalized crossed products

Let

$$\begin{array}{rccc} \Theta : & G & \longrightarrow & \mathbf{PicS}(R), \\ & x & \longmapsto & [\Theta_x], \end{array}$$

be a unital partial representation with $\Theta_x \otimes \Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$. By (15) there is a family of R-bimodule isomorphisms

$$f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x \Theta_{xy},\ x, y \in G\}. \tag{31}$$

Following [23], we say that f^Θ is a *factor set* for Θ if f^Θ satisfies the following commutative diagram:

$$\begin{tikzcd}
\Theta_x \otimes_R \Theta_y \otimes_R \Theta_z \arrow[r, "\Theta_x \otimes f^\Theta_{y,z}"] \arrow[d, "f^\Theta_{x,y}\otimes \Theta_z"'] & \Theta_x \otimes_R 1_y\Theta_{yz} \arrow[r, equal] & 1_{xy}\Theta_x \otimes_R \Theta_{yz} \arrow[d, "f^\Theta_{x,yz}"] \\
1_x\Theta_{xy} \otimes_R \Theta_z \arrow[rr, "f^\Theta_{xy,z}"'] & & 1_x 1_{xy}\Theta_{xyz}
\end{tikzcd} \tag{32}$$

Let $f^\Theta = \{f^\Theta_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ be a factor set for Θ. The set $\Delta(\Theta) = \bigoplus_{x\in G} \Theta_x$ with multiplication defined by

$$u_x \overset{\Theta}{\circ} u_y = f^\Theta_{x,y}(u_x \otimes u_y) \in 1_x\Theta_{xy}, \quad u_x \in \Theta_x, u_y \in \Theta_y, \tag{33}$$

is called a *partial generalized crossed product*.

Proposition 3.13. *Let $\Theta : G \longrightarrow \mathbf{PicS}(R)$ be a unital partial representation with $\varepsilon_x = \Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$, and let $f^\Theta = \{f^\Theta_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ be a factor set for Θ. Then, the partial generalized crossed product $\Delta(\Theta)$ is an associative ring with unity and $R \simeq \Theta_1$ is a subring of $\Delta(\Theta)$.*

Proof. The commutativity of (32) implies the associativity of the multiplication in $\Delta(\Theta)$. It is easy to see that Θ_1 is a subring of $\Delta(\Theta)$ since $f^\Theta_{1,1} : \Theta_1 \otimes \Theta_1 \to \Theta_1$ is an R-bimodule isomorphism. Let $j : R \longrightarrow \Theta_1$ be a R-bimodule isomorphism; it exists because Θ is a partial representation and $[\Theta_1] = [R]$. Write $v = j(1)$. Then $\Theta_1 = j(R) = Rv = vR$ and

$$rv = rj(1) = j(r) = j(1)r = vr, \quad \text{for all } r \in R.$$

Let $c \in R$ be such that $f^\Theta_{1,1}(v \otimes v) = cv$. It is easy to see that $c \in \mathcal{U}(\mathcal{Z})$. Denote $u = c^{-1}v \in \Theta_1$. Then $ru = rc^{-1}v = c^{-1}vr = ur$, for all $r \in R$ and

$$f^\Theta_{1,1}(u \otimes u) = f^\Theta_{1,1}(c^{-1}v \otimes c^{-1}v) = c^{-1}f^\Theta_{1,1}(v \otimes v)c^{-1} = c^{-1}cvc^{-1} = vc^{-1} = u. \tag{34}$$

Let $\iota : R \longrightarrow \Theta_1$ be the R-bimodule isomorphism defined by $\iota(r) = ru$. Given $r, r' \in R$ we have

$$\iota(r) \overset{\Theta}{\circ} \iota(r') = f^\Theta_{1,1}(ru \otimes r'u) = rr'f^\Theta_{1,1}(u \otimes u) \overset{(34)}{=} rr'u = \iota(rr'),$$

so, ι is a ring isomorphism.

Finally, we check that $\iota(1) = u \in \Theta_1$ is the unity of $\Delta(\Theta)$. Let $u_x \in \Theta_x$, $x \in G$. Since $f^\Theta_{1,x} : \Theta_1 \otimes_R \Theta_x \longrightarrow \Theta_x$ is an R-bimodule isomorphism, there is $\sum_{i=1}^n ur_i \otimes u^i_x = u \otimes \widetilde{u}_x \in \Theta_1 \otimes_R \Theta_x$, where $\widetilde{u}_x = \sum_{i=1}^n r_iu^i_x$, such that $f^\Theta_{1,x}(u \otimes \widetilde{u}_x) = u_x$. Then,

$$(u \overset{\Theta}{\circ} u_x) = f^\Theta_{1,x}(u \otimes u_x) = f^\Theta_{1,x}(u \otimes f^\Theta_{1,x}(u \otimes \widetilde{u}_x)) = f^\Theta_{1,x}(f^\Theta_{1,1}(u \otimes u) \otimes \widetilde{u}_x) \overset{(34)}{=} f^\Theta_{1,x}(u \otimes \widetilde{u}_x) = u_x.$$

Analogously, since $f^\Theta_{x,1} : \Theta_x \otimes_R \Theta_1 \longrightarrow \Theta_x$ is an isomorphism, we have $(u_x \overset{\Theta}{\circ} u) = u_x$, for all $u_x \in \Theta_x$. Therefore, u is the unity of $\Delta(\Theta)$. □

We identify R with Θ_1 and 1 with $\iota(1) = u \in \Theta_1$, via the ring isomorphism ι. Then, $R \subseteq \Delta(\Theta)$ is an extension of rings with the same unity.

Remark 3.14. Let $\Delta(\Theta)$ be a partial generalized crossed product, $\iota : R \longrightarrow \Theta_1$ an isomorphism of rings and R-bimodules and $\iota(1) = u \in \Theta_1$ be the unity of $\Delta(\Theta)$. Then, the following diagrams are commutative:

$$\begin{tikzcd} R\otimes_R \Theta_x \arrow[rr, "\simeq"] \arrow[dr, "\iota\otimes\Theta_x"'] & & \Theta_x \\ & \Theta_1\otimes_R\Theta_x \arrow[ur, "f^{\Theta}_{1,x}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Theta_x\otimes_R R \arrow[rr, "\simeq"] \arrow[dr, "\Theta_x\otimes\iota"'] & & \Theta_x \\ & \Theta_x\otimes_R\Theta_1 \arrow[ur, "f^{\Theta}_{x,1}"'] & \end{tikzcd} \tag{35}$$

Indeed, given $r \in R$ and $u_x \in \Theta_x$, we have

$$(f^{\Theta}_{1,x} \circ (\iota \otimes \Theta_x))(r \otimes u_x) = f^{\Theta}_{1,x}(\iota(r) \otimes u_x) = r f^{\Theta}_{1,x}(\iota(1) \otimes u_x) = r u_x.$$

The second diagram is analogous. By (35), we can write

$$u_x \overset{\Theta}{\circ} r = u_x r \quad \text{and} \quad r \overset{\Theta}{\circ} u_x = r u_x, \ \forall\ u_x \in \Theta_x \text{ and } r \in R. \tag{36}$$

Proposition 3.15. *Let $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ be a family of R-bimodule isomorphisms which satisfies the commutative diagrams (35). Given $x, y, z \in G$, define an R-bimodule isomorphism $\beta_{x,y,z} : 1_x1_{xy}\Theta_{xyz} \longrightarrow 1_x1_{xy}\Theta_{xyz}$ via the commutative diagram*

$$\begin{tikzcd} \Theta_x\otimes_R\Theta_y\otimes_R\Theta_z \arrow[rr, "f^{\Theta}_{x,y}\otimes\Theta_z"] \arrow[d, "\Theta_x\otimes f^{\Theta}_{y,z}"'] & & 1_x\Theta_{xy}\otimes_R\Theta_z \arrow[d, "f^{\Theta}_{xy,z}"] \\ 1_{xy}\Theta_x\otimes_R\Theta_{yz} \arrow[dr, "f^{\Theta}_{x,yz}"'] & & 1_x1_{xy}\Theta_{xyz} \\ & 1_x1_{xy}\Theta_{xyz} \arrow[ur, dashed, "\beta_{x,y,z}"'] & \end{tikzcd}$$

that is,

$$\beta_{x,y,z} \circ f^{\Theta}_{x,yz} \circ (\Theta_x \otimes f^{\Theta}_{y,z}) = f^{\Theta}_{xy,z} \circ (f^{\Theta}_{x,y} \otimes \Theta_z),\ \forall\ x, y, z \in G. \tag{37}$$

Then,

$$\widetilde{\beta_{-,-,-}} : \begin{array}{ccc} G\times G\times G & \longrightarrow & \mathcal{U}(\mathcal{Z}) \\ (x,y,z) & \longmapsto & \widetilde{\beta_{x,y,z}} \end{array},$$

is a normalized element of $Z^3_{\Theta}(G, \alpha, \mathcal{Z})$, where $\widetilde{\beta_{x,y,z}}$ is defined as in Lemma 3.10. Furthermore, if $\widetilde{\beta_{-,-,-}} \in B^3_{\Theta}(G, \alpha, \mathcal{Z})$, that is, if there exists $\sigma_{-,-} : G \times G \longrightarrow \mathcal{U}(\mathcal{Z})$, with $\sigma_{x,y} \in \mathcal{U}(\mathcal{Z}1_x1_{xy})$ and

$$\widetilde{\beta_{x,y,z}} = \alpha_x(\sigma_{y,z}1_{x^{-1}})\sigma^{-1}_{xy,z}\sigma_{x,yz}\sigma^{-1}_{x,y}, \quad \textit{for all } x, y, z \in G, \tag{38}$$

then the family of R-bimodule isomorphisms

$$\bar{f}^{\Theta}_{x,y} : \begin{array}{ccc} \Theta_x\otimes_R\Theta_y & \longrightarrow & 1_x\Theta_{x,y}, \\ u_x\otimes u_y & \longmapsto & \sigma_{x,y}f^{\Theta}_{x,y}(u_x\otimes u_y) \end{array}$$

is a factor set for Θ.

Proof. Since $\beta_{x,y,z} \in \mathbf{Aut}_{R-R}(1_x 1_{xy}\Theta_{xyz})$, Lemma 3.10 implies that $\widetilde{\beta_{x,y,z}} \in \mathcal{U}(\mathcal{Z}1_x 1_{xy}1_{xyz})$, for all $x, y, z \in G$. Write $f^{\Theta}_{x,y}(u_x \otimes u_y) = (u_x \overset{\Theta}{\circ} u_y)$. By (37) we have that $\beta_{x,y,z}(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} u_z)) = ((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z)$, for all $u_x \in \Theta_x, u_y \in \Theta_y$ and $u_z \in \Theta_z$. Since $(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} u_z)) \in 1_x 1_{xy}\Theta_{xyz}$, then by (25),

$$\widetilde{\beta_{x,y,z}}(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} u_z)) = ((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z). \tag{39}$$

Let $x, y, z, t \in G$ e $u_x \in \Theta_x, u_y \in \Theta_y, u_z \in \Theta_z$ and $u_t \in \Theta_t$. Then,

$$\begin{aligned}
(((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \overset{\Theta}{\circ} u_t) &\overset{(39)}{=} \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}(u_x \overset{\Theta}{\circ} \widetilde{\beta_{y,z,t}}(u_y \overset{\Theta}{\circ} (u_z \overset{\Theta}{\circ} u_t)))\\
&\overset{(28)}{=} \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} (u_z \overset{\Theta}{\circ} u_t)))\\
&\overset{(39)}{=} \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}(((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \overset{\Theta}{\circ} u_t).
\end{aligned}$$

Thus,

$$(((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \overset{\Theta}{\circ} u_t) = \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}(((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \overset{\Theta}{\circ} u_t),$$

for all $u_x \in \Theta_x, u_y \in \Theta_y, u_z \in \Theta_z$ and $u_t \in \Theta_z$. Write

$$1_x = \sum_{(x)}(\omega_x \overset{\Theta}{\circ} \omega_{x^{-1}}), 1_y = \sum_{(y)}(\omega_y \overset{\Theta}{\circ} \omega_{y^{-1}}), 1_z = \sum_{(z)}(\omega_z \overset{\Theta}{\circ} \omega_{z^{-1}}) \text{ and } 1_t = \sum_{(t)}(\omega_t \overset{\Theta}{\circ} \omega_{t^{-1}}),$$

where $\omega_l \in \Theta_l$, for all $l \in G$. Then,

$$(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z) \overset{\Theta}{\circ} \omega_t) = \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z) \overset{\Theta}{\circ} \omega_t).$$

Applying $f^{\Theta}_{xyzt,t^{-1}}$ and using (39) we obtain:

$$\begin{aligned}
&\widetilde{\beta_{xyz,t,t^{-1}}}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z) \overset{\Theta}{\circ} (\omega_t \overset{\Theta}{\circ} \omega_{t^{-1}}))\\
&= \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}\widetilde{\beta_{xyz,t,t^{-1}}}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z) \overset{\Theta}{\circ} (\omega_t \overset{\Theta}{\circ} \omega_{t^{-1}})).
\end{aligned}$$

Multiplying by $\widetilde{\beta_{xyz,t,t^{-1}}}^{-1}$, summing over (t), using the commutativity of (35) and (17), we compute:

$$\begin{aligned}
1_{xyz}1_{xyzt}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z))1_t &= \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z))1_t\\
1_{xyz}1_{xyzt}1_{xyzt}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z)) &= \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}1_{xyzt}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z))\\
1_{xyz}1_{xyzt}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z)) &= \widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1}(((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} \omega_z)).
\end{aligned}$$

Repeating the same argument for z, y, x, we get:

$$\widetilde{\beta_{x,y,z}}\widetilde{\beta_{x,yz,t}}\alpha_x(\widetilde{\beta_{y,z,t}}1_{x^{-1}})\widetilde{\beta_{x,y,zt}}^{-1}\widetilde{\beta_{xy,z,t}}^{-1} = 1_x 1_{xy}1_{xyz}1_{xyzt},$$

for all $x, y, z, t \in G$. Hence, $\widetilde{\beta_{-,-,-}} \in Z^3_{\Theta}(G, \alpha, \mathcal{Z})$.

Let us check that $\widetilde{\beta_{-,-,-}}$ is normalized. Taking $x = 1$, we obtain using (39) and the commutativity of the diagrams in (35), that $\widetilde{\beta_{1,y,z}}(ru_y \overset{\Theta}{\circ} u_z) = (ru_y \overset{\Theta}{\circ} u_z)$, for all $r \in R$, $u_x \in \Theta_x$ and $u_y \in \Theta_y$. Putting $r = 1$ and following the above argument, we conclude that $\widetilde{\beta_{1,y,z}} = 1_y 1_{yz}$. Analogously, we obtain

$$\widetilde{\beta_{x,1,z}} = 1_x 1_{xz} \ \text{ e } \ \widetilde{\beta_{x,y,1}} = 1_x 1_{xy}, \ \text{for all } x, y, z \in G.$$

Therefore, $\widetilde{\beta_{-,-,-}} \in Z^3_\Theta(G, \alpha, \mathcal{Z})$ is normalized.

We now verify that $\bar{f}^\Theta = \{\bar{f}^\Theta_{x,y}, \ x, y \in G\}$ is a factor set for Θ. Let $x, y, z \in G$ and $u_x \in \Theta_x, u_y \in \Theta_y, u_z \in \Theta_z$. Then,

$$\begin{aligned}
\bar{f}^\Theta_{x,yz}(u_x \otimes \bar{f}^\Theta_{y,z}(u_y \otimes u_z)) &= \sigma_{x,yz}(u_x \overset{\Theta}{\circ} \sigma_{y,z}(u_y \overset{\Theta}{\circ} u_z)) \\
&\overset{(28)}{=} \sigma_{x,yz}\alpha_x(\sigma_{y,z}1_{x^{-1}})(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} u_z)) \\
&= \sigma_{x,yz}\alpha_x(\sigma_{y,z}1_{x^{-1}})\widetilde{\beta_{x,y,z}}((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \\
&\overset{(38)}{=} \sigma_{xy,z}\sigma_{x,y}((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \\
&= \bar{f}^\Theta_{xy,z}(\bar{f}^\Theta_{x,y}(u_x \otimes u_y) \otimes u_z).
\end{aligned}$$

Consequently, $\bar{f}^\Theta = \{\bar{f}^\Theta_{x,y}, \ x, y \in G\}$ is a factor set for Θ. $\square$

The next fact follows from the proof of Proposition 3.15 and (29).

Corollary 3.16. *Let $\Gamma : G \longrightarrow \mathbf{PicS}(R)$ be a unital partial representation with $\Gamma_x \otimes_R \Gamma_{x^{-1}} \simeq R1_x$ and $\Gamma_x|\Theta_x$, for all $x \in G$. Let, furthermore, $f^\Gamma = \{f^\Gamma_{x,y} : \Gamma_x \otimes \Gamma_y \longrightarrow 1_x\Gamma_{xy}, \ \in x, y \in G\}$ be a family of R-bimodule isomorphisms which satisfies the commutative diagrams in (35) and let $\beta^\Gamma_{x,y,z} : 1_x 1_{xy}\Theta_{xyz} \longrightarrow 1_x 1_{xy}\Theta_{xyz}$ be an R-bimodule isomorphism such that*

$$\beta^\Gamma_{x,y,z} \circ f^\Gamma_{x,yz} \circ (\Gamma_x \otimes f^\Gamma_{y,z}) = f^\Gamma_{xy,z} \circ (f^\Gamma_{x,y} \otimes \Gamma_z), \ \forall \ x, y, z \in G.$$

Then $\widetilde{\beta^\Gamma_{-,-,-}} \in Z^3_\Theta(G, \alpha, \mathcal{Z})$.

The next result establishes the uniqueness of the class in $H^3_\Theta(G, \alpha, \mathcal{Z})$ of the 3-cocycle given in Corollary 3.16.

Proposition 3.17. *Let $\Gamma : G \longrightarrow \mathbf{PicS}(R)$ be a unital partial representation with $\Gamma_x \otimes_R \Gamma_{x^{-1}} \simeq R1_x$ and $\Gamma_x|\Theta_x$, for all $x \in G$. Let, furthermore, $f^\Gamma = \{f^\Gamma_{x,y} : \Gamma_x \otimes_R \Gamma_y \longrightarrow 1_x\Gamma_{xy}, \ x, y \in G\}$ be a family of R-bimodule isomorphisms which satisfies the commutative diagrams in (35). Then the class $[\widetilde{\beta^\Gamma_{-,-,-}}]$ in $H^3_\Theta(G, \alpha, \mathcal{Z})$ does not depend on the choice of the representatives in $[\Gamma_x]$, nor on the choice of R-bimodule isomorphisms f^Γ.*

Proof. Let $\Omega_x \in [\Gamma_x]$. Then there exists an R-bimodule isomorphism $a_x : \Omega_x \longrightarrow \Gamma_x$, for each $x \in G$. Let

$$f^\Omega = \{f^\Omega_{x,y} : \Omega_x \otimes_R \Omega_y \longrightarrow 1_x\Omega_{xy}, \ x, y \in G\}, \quad f^\Gamma = \{f^\Gamma_{x,y} : \Gamma_x \otimes_R \Gamma_y \longrightarrow 1_x\Gamma_{xy}, \ x, y \in G\}$$

be families of R-module isomorphisms which satisfy the commutative diagrams in (35), and let $\widetilde{\beta^\Omega_{-,-,-}}$ and $\widetilde{\beta^\Gamma_{-,-,-}}$ be the 3-cocycles associated with f^Ω and f^Γ, respectively. By definition, we have that

$$\beta^\Omega_{x,y,z} \circ f^\Omega_{x,yz} \circ (\Omega_x \otimes f^\Omega_{y,z}) = f^\Omega_{xy,z} \circ (f^\Omega_{x,y} \otimes \Omega_z) \quad \text{and} \quad \beta^\Gamma_{x,y,z} \circ f^\Gamma_{x,yz} \circ (\Gamma_x \otimes f^\Gamma_{y,z}) = f^\Gamma_{xy,z} \circ (f^\Gamma_{x,y} \otimes \Gamma_z).$$

Denote by $b_{x,y} : 1_x\Omega_{xy} \longrightarrow 1_x\Omega_{xy}$ the R-bimodule isomorphism defined by the commutative diagram

$$\begin{tikzcd}
\Omega_x \otimes_R \Omega_y \arrow[rr, "f^{\Omega}_{x,y}"] \arrow[d, "a_x\otimes a_y"'] & & 1_x\Omega_{xy} \arrow[d, dashed, "b_{x,y}"] \\
\Gamma_x \otimes_R \Gamma_y \arrow[dr, "f^{\Gamma}_{x,y}"'] & & 1_x\Omega_{xy} \\
& 1_x\Gamma_{xy} \arrow[ur, "a_{xy}^{-1}"'] &
\end{tikzcd}$$

that is,

$$a_{xy}\circ b_{x,y}\circ f^{\Omega}_{x,y} = f^{\Gamma}_{x,y}\circ(a_x\otimes a_y), \quad \forall\, x,y\in G.$$

Let $\widetilde{b_{x,y}}\in\mathcal{U}(\mathcal{Z}1_x1_{xy})$ be the image of $b_{x,y}$ under the isomorphism of Lemma 3.10. By (25) we obtain that $\widetilde{b_{x,y}}t_{xy} = b_{x,y}(1_xt_{xy})$, for all $t_{xy}\in\Omega_{xy}$. Writing $f^{\Omega}_{x,y}(t_x\otimes t_y) = (t_x \overset{\Omega}{\circ} t_y)$ and $f^{\Gamma}_{x,y}(v_x\otimes v_y) = (v_x \overset{\Gamma}{\circ} v_y)$, we have:

$$\widetilde{\beta^{\Omega}_{x,y,z}}(t_x \overset{\Omega}{\circ} (t_y \overset{\Omega}{\circ} t_z)) = ((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z), \tag{40}$$

$$\widetilde{\beta^{\Gamma}_{x,y,z}}(v_x \overset{\Gamma}{\circ} (v_y \overset{\Gamma}{\circ} v_z)) = ((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z), \tag{41}$$

$$\widetilde{b_{x,y}}a_{xy}(t_x \overset{\Omega}{\circ} t_y) = (a_x(t_x) \overset{\Gamma}{\circ} a_y(t_y)). \tag{42}$$

Given $x,y,z\in G$ and $t_x\in\Omega_x, t_y\in\Omega_y, t_z\in\Omega_z$ we compute:

$$\begin{aligned}
a_{xyz}((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z) &\overset{(42)}{=} \widetilde{b^{-1}_{xy,z}}(a_{xy}(t_x \overset{\Omega}{\circ} t_y) \overset{\Gamma}{\circ} a_z(t_z))\\
&\overset{(42)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}((a_x(t_x) \overset{\Gamma}{\circ} a_y(t_y)) \overset{\Gamma}{\circ} a_z(t_z))\\
&\overset{(41)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}(a_x(t_x) \overset{\Gamma}{\circ} (a_y(t_y) \overset{\Gamma}{\circ} a_z(t_z)))\\
&\overset{(42)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}(a_x(t_x) \overset{\Gamma}{\circ} \widetilde{b_{y,z}}a_{yz}(t_y \overset{\Omega}{\circ} t_z))\\
&\overset{(28)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})(a_x(t_x) \overset{\Gamma}{\circ} a_{yz}(t_y \overset{\Omega}{\circ} t_z))\\
&\overset{(42)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}}a_{xyz}(t_x \overset{\Omega}{\circ} (t_y \overset{\Omega}{\circ} t_z))\\
&\overset{(40)}{=} \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}}\widetilde{\beta^{\Omega}_{x,y,z}}^{-1} a_{xyz}((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z)\\
&= a_{xyz}(\widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}}\widetilde{\beta^{\Omega}_{x,y,z}}^{-1}((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z)).
\end{aligned}$$

Since a_{xyz} is an isomorphism, it follows that

$$((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z) = \widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}}\widetilde{\beta^{\Omega}_{x,y,z}}^{-1}((t_x \overset{\Omega}{\circ} t_y) \overset{\Omega}{\circ} t_z),$$

for all $t_x\in\Omega_x, t_y\in\Omega_y, t_z\in\Omega_z$. Using the same argument of [24, Lemma 4.3], we conclude:

$$\widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\widetilde{\beta^{\Gamma}_{x,y,z}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}}\widetilde{\beta^{\Omega}_{x,y,z}}^{-1} = 1_x1_{xy}1_{xyz}.$$

Hence, $\widetilde{\beta^{\Omega}_{x,y,z}} = \widetilde{\beta^{\Gamma}_{x,y,z}}\widetilde{b^{-1}_{xy,z}}\widetilde{b^{-1}_{x,y}}\alpha_x(\widetilde{b_{y,z}}1_{x^{-1}})\widetilde{b_{x,yz}} = \widetilde{\beta^{\Gamma}_{x,y,z}}(\delta^2\widetilde{b_{-,-}})(x,y,z)$, where $\widetilde{b_{-,-}}: G\times G\to\mathcal{U}(\mathcal{Z})$ and $\widetilde{b_{x,y}}\in\mathcal{U}(\mathcal{Z}1_x1_{xy})$. Consequently, $[\widetilde{\beta^{\Omega}_{-,-,-}}] = [\widetilde{\beta^{\Gamma}_{-,-,-}}]$ in $H^3_{\Theta}(G,\alpha,\mathcal{Z})$. $\square$

We proceed with the following technical fact.

Lemma 3.18. *Let $\Theta, \Omega, \Gamma : G \longrightarrow \mathbf{PicS}(R)$ be unital partial representations with $\Gamma_x|\Theta_x$, $\Omega_x|\Theta_x$ and $\Omega_x \otimes_R \Omega_{x^{-1}} \simeq \Gamma_x \otimes_R \Gamma_{x^{-1}} \simeq \Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$. Let, furthermore, $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ be a factor set for Θ, let*

$$f^{\Gamma} = \{f^{\Gamma}_{x,y} : \Gamma_x \otimes_R \Gamma_y \longrightarrow 1_x\Gamma_{xy},\ x, y \in G\},\ f^{\Omega} = \{f^{\Omega}_{x,y} : \Omega_x \otimes_R \Omega_y \longrightarrow 1_x\Omega_{xy},\ x, y \in G\}$$

be families of R-bimodule isomorphisms which satisfy the commutative diagrams in (35) and let $\widetilde{\beta^{\Gamma}_{-,-,-}}$, and $\widetilde{\beta^{\Omega}_{-,-,-}}$ be the 3-cocycles in $Z^3_{\Theta}(G, \alpha, \mathcal{Z})$ associated to f^{Γ} and f^{Ω}, respectively. Then,

$$\begin{array}{rccl} \Lambda : & G & \longrightarrow & \mathbf{PicS}(R), \\ & x & \longmapsto & [\Gamma_x \otimes_R \Theta_{x^{-1}} \otimes_R \Omega_x] \end{array}$$

is a unital partial representation with $\Lambda_x \otimes_R \Lambda_{x^{-1}} \simeq R1_x$ and $\Lambda_x|\Theta_x$, for all $x \in G$. The 3-cocycle associated with the family of R-bimodule isomorphisms $f^{\Lambda} = \{f^{\Lambda}_{x,y} : \Lambda_x \otimes_R \Lambda_y \longrightarrow 1_x\Lambda_{xy},\ x, y \in G\}$ defined by

$$\begin{tikzcd}
\Gamma_x \otimes_R \Theta_{x^{-1}} \otimes_R \Omega_x \otimes_R \Gamma_y \otimes_R \Theta_{y^{-1}} \otimes_R \Omega_y \arrow[dr, "\Gamma_x \otimes T_{\Theta_{x^{-1}} \otimes \Omega_x, \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Omega_y}"] \arrow[dddr, dashed, bend right, "f^{\Lambda}_{x,y}"'] & \\
& \Gamma_x \otimes_R \Gamma_y \otimes_R \Theta_{y^{-1}} \otimes_R \Theta_{x^{-1}} \otimes_R \Omega_x \otimes_R \Omega_y \arrow[d, "f^{\Gamma}_{x,y} \otimes f^{\Theta}_{y^{-1},x^{-1}} \otimes f^{\Omega}_{x,y}"] \\
& 1_x\Gamma_{xy} \otimes_R 1_{y^{-1}}\Theta_{(xy)^{-1}} \otimes_R 1_x\Omega_{xy} \arrow[d, equal] \\
& 1_x\Gamma_{xy} \otimes_R \Theta_{(xy)^{-1}} \otimes_R \Omega_{xy},
\end{tikzcd}$$

is given by $\widetilde{\beta^{\Lambda}_{-,-,-}} = \widetilde{\beta^{\Gamma}_{-,-,-}}\widetilde{\beta^{\Omega}_{-,-,-}}$, where $T_{-,-}$ is the isomorphism from Proposition 2.10. If f^{Γ} and f^{Ω} are factor sets for Γ and Ω, respectively, then f^{Λ} is a factor set for Λ and $\Delta(\Lambda)$ is a generalized partial crossed product.

Proof. Clearly $\Lambda_1 = R$. By the above diagram there is an R-bimodule isomorphism $\Lambda_x \otimes_R \Lambda_y \simeq R1_x \otimes_R \Lambda_{xy}$, where $x, y \in G$. In particular, $\Lambda_x \otimes_R \Lambda_{x^{-1}} \simeq R1_x$. Moreover, it is easy to see that $\Lambda_x \otimes_R R1_{x^{-1}} \simeq \Lambda_x \simeq R1_x \otimes_R \Lambda_x$ and $\Lambda_x \otimes_R R1_y \simeq R1_{xy}\Lambda_x$, for all $x, y \in G$. Then,

$$\Lambda_{x^{-1}} \otimes_R \Lambda_x \otimes_R \Lambda_y \simeq \Lambda_{x^{-1}} \otimes_R R1_x \otimes_R \Lambda_{xy} \simeq \Lambda_{x^{-1}} \otimes_R \Lambda_{xy}.$$

On the other hand,

$$\Lambda_x \otimes_R \Lambda_y \otimes_R \Lambda_{y^{-1}} \simeq R1_x \otimes_R \Lambda_{xy} \otimes_R \Lambda_{y^{-1}} \simeq \Lambda_{xy} \otimes_R R1_{y^{-1}} \otimes_R \Lambda_{y^{-1}} \simeq \Lambda_{xy} \otimes_R \Lambda_{y^{-1}}.$$

Thus, Λ is a unital partial representation. Since $\Gamma_x|\Theta_x$ and $\Omega_x|\Theta_x$, then $\Lambda_x \simeq \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x|\Theta_x$, for all $x \in G$.

By definition,

$$\widetilde{\beta^{\Gamma}_{x,y,z}}(v_x \overset{\Gamma}{\circ} (v_y \overset{\Gamma}{\circ} v_z)) = ((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z) \quad \text{and} \quad \widetilde{\beta^{\Omega}_{x,y,z}}(w_x \overset{\Omega}{\circ} (w_y \overset{\Omega}{\circ} w_z)) = ((w_x \overset{\Omega}{\circ} w_y) \overset{\Omega}{\circ} w_z),$$

for all $v_i \in \Gamma_i$ and $w_i \in \Omega_i$, $i \in G$. Using [28, Lemma 1.4] it can be seen that

$$\beta^{\Gamma}_{x,y,z}\beta^{\Omega}_{x,y,z} \circ f^{\Lambda}_{x,yz} \circ (\Lambda_x \otimes f^{\Lambda}_{y,z}) = f^{\Lambda}_{xy,z} \circ (f^{\Lambda}_{x,y} \otimes \Lambda_z),\ x, y, z \in G.$$

In particular, if f^{Γ} is a factor set for Γ and f^{Ω} is a factor set for Ω, then β^{Γ} and β^{Ω} are trivial, thus $\beta^{\Gamma}\beta^{\Omega}$ is trivial too. Therefore f^{Λ} is a factor set for Λ. $\square$

Definition 3.19. Let $\Delta(\Theta)$ and $\Delta(\Gamma)$ be partial generalized crossed products with R-bimodule and ring isomorphism $i : R \longrightarrow \Theta_1$ and $j : R \longrightarrow \Gamma_1$, respectively, as in Remark 3.14. A morphism of partial generalized crossed products $F : \Delta(\Theta) \longrightarrow \Delta(\Gamma)$ is a set of R-bimolude morphisms $\{F_x : \Theta_x \longrightarrow \Gamma_x, x \in G\}$ such that $F_1 \circ i = j$ and the following commutative diagram is satisfied:

$$\begin{tikzcd}
\Theta_x \otimes_R \Theta_y \arrow[r, "f^{\Theta}_{x,y}"] \arrow[d, "F_x \otimes F_y"'] & 1_x\Theta_{xy} \arrow[d, "F_{xy}"] \\
\Gamma_x \otimes_R \Gamma_y \arrow[r, "f^{\Gamma}_{x,y}"'] & 1_x\Gamma_{xy}
\end{tikzcd} \tag{43}$$

A morphism F of partial generalized crossed products is called an isomorphism if each morphism $F_x : \Theta_x \longrightarrow \Gamma_x$ is an R-bimodule isomorphism.

We denote by $[\Delta(\Omega)]$ the isomorphism class of the partial generalized crossed product $\Delta(\Omega)$.

Remark 3.20. If $F = \{F_x : \Theta_x \longrightarrow \Gamma_x,\ x \in G\}$ is a family of R-bimodule isomorphism satisfying the commutative diagram (43), then F is an isomorphism of partial generalized crossed products. Indeed, it is enough to show that $F_1 \circ i = j$. It follows from Remark 3.14 and the commutative diagram (43) that the following diagram is commutative:

$$\begin{tikzcd}
R \otimes_R \Theta_x \arrow[r, "i\otimes\Theta_x"] \arrow[dr, "\simeq"'] & \Theta_1 \otimes_R \Theta_x \arrow[r, "F_1\otimes F_x"] \arrow[d, "f^{\Theta}_{1,x}"'] & \Gamma_1 \otimes_R \Gamma_x \arrow[d, "f^{\Gamma}_{1,x}"] & R \otimes_R \Gamma_x \arrow[l, "\simeq"'] \arrow[dl, "j\otimes\Gamma_x"] \\
& \Theta_x \arrow[r, "F_x"'] & \Gamma_x &
\end{tikzcd}$$

Given $v_x \in \Gamma_x$, there is $u_x \in \Theta_x$ such that $F_x(u_x) = v_x$. Then $(F_1(i(1)) \overset{\Gamma}{\circ} F_x(u_x)) = F_x(u_x)$, that is,

$$(F_1(i(1)) \overset{\Gamma}{\circ} v_x) = v_x, \quad \text{for all} \quad v_x \in \Gamma_x.$$

On the other hand, the commutative diagram

$$\begin{tikzcd}
\Theta_x \otimes_R R \arrow[r, "\Theta_x\otimes i"] \arrow[dr, "\simeq"'] & \Theta_x \otimes_R \Theta_1 \arrow[r, "F_x\otimes F_1"] \arrow[d, "f^{\Theta}_{x,1}"'] & \Gamma_x \otimes_R \Gamma_1 \arrow[d, "f^{\Gamma}_{x,1}"] & \Gamma_x \otimes_R R \arrow[l, "\Gamma_x\otimes j"'] \arrow[dl, "\simeq"] \\
& \Theta_x \arrow[r, "F_x"'] & \Gamma_x &
\end{tikzcd}$$

implies that $(v_x \overset{\Gamma}{\circ} F_1(i(1))) = v_x$, for all $v_x \in \Gamma_x$. Thus, $F(i(1)) = j(1)$ is the unity of $\Delta(\Gamma)$. Therefore, $F \circ i = j$.

Remark 3.21. Let Θ be a partial representation and $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ a factor set for Θ. For each $x \in G$, we have an R-bimodule isomorphism $f^{\Theta}_{x,x-1} : \Theta_x \otimes_R \Theta_{x-1} \longrightarrow R1_x$, whose inverse is

$$\begin{array}{rccl} (f^{\Theta}_{x,x^{-1}})^{-1}: & R1_x & \longrightarrow & \Theta_x \otimes_R \Theta_{x^{-1}} \\ & r1_x & \longrightarrow & \displaystyle\sum_{(x)} r\omega_x \otimes \omega_{x^{-1}}, \end{array}$$

where $\displaystyle\sum_{(x)}(\omega_x \overset{\Theta}{\circ} \omega_{x^{-1}}) = 1_x$. Restricting, we obtain an R-bimodule isomorphism $1_x f^{\Theta}_{xy,(xy)^{-1}} : 1_x\Theta_{xy} \otimes_R \Theta_{(xy)^{-1}} \longrightarrow R1_x1_{xy}$. Denote $1_x = \displaystyle\sum_{(x)}(\omega_x \overset{\Theta}{\circ} \omega_{x^{-1}})$ and $1_y = \displaystyle\sum_{(y)}(\omega_y \overset{\Theta}{\circ} \omega_{y^{-1}})$. Then, by associativity, we have

$$\begin{aligned} \sum_{(x),(y)} \Big((\omega_x \overset{\Theta}{\circ} \omega_y) \overset{\Theta}{\circ} (\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}})\Big) &= \sum_{(x),(y)} \Big(\omega_x \overset{\Theta}{\circ} (\omega_y \overset{\Theta}{\circ} \omega_{y^{-1}}) \overset{\Theta}{\circ} \omega_{x^{-1}}\Big) \\ &= \sum_{(x)}(\omega_x 1_y \overset{\Theta}{\circ} \omega_{x^{-1}}) = \sum_{(x)} 1_{xy}(\omega_x \overset{\Theta}{\circ} \omega_{x^{-1}}) = 1_{xy}1_x. \end{aligned}$$

Thus,

$$\begin{array}{rccl} (1_x f^{\Theta}_{xy,y^{-1}x^{-1}})^{-1}: & R1_x1_{xy} & \longrightarrow & 1_x\Theta_{xy} \otimes_R \Theta_{(xy)^{-1}} \\ & r1_x1_{xy} & \longmapsto & \displaystyle\sum_{(x),(y)} r((\omega_x \overset{\Theta}{\circ} \omega_y) \otimes (\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}})). \end{array}$$

3.4. Unital partial representations $G \to \mathcal{S}_R(S)$

As we have seen in Section 3.3, if $\Theta : G \longrightarrow \mathbf{PicS}(R)$ is a unital partial representation such that we have a generalized partial crossed product $\Delta(\Theta)$, then $R \subseteq \Delta(\Theta)$ is a ring extension with the same unity. In this section we shall see how to obtain a generalized partial crossed product from a ring extension and a certain partial representation.

Let $R \subseteq S$ be a ring extension with the same unity element. Denote by $\mathcal{S}_R(S)$ the set of the R-subbimodules of S. Given $M, N \in \mathcal{S}_R(S)$, define the product:

$$MN = \left\{\sum_{i=1}^{k} m_i n_i;\ m_i \in M, n_i \in N\right\}.$$

With the above operation $\mathcal{S}_R(S)$ is a monoid with neutral element R.

A partial representation

$$\begin{array}{rccl} \Theta: & G & \longrightarrow & \mathcal{S}_R(S), \\ & x & \longmapsto & \Theta_x \end{array}$$

will be called *unital* if $\varepsilon_x = \Theta_x\Theta_{x^{-1}} = R1_x$, where 1_x is a central idempotent in R, for each $x \in G$. The equality $\Theta_x\Theta_{x^{-1}}\Theta_x = \Theta_x$, $(x \in G)$, implies that $R1_x\Theta_x = \Theta_x$ and $\Theta_x R1_{x^{-1}} = \Theta_x$. Hence $1_x u_x = u_x$ and $u_x 1_{x^{-1}} = u_x$, for all $u_x \in \Theta_x$ and $x \in G$. Furthermore, if M is an S-bimodule then $\Theta_x M = 1_x M = \{m \in M, 1_x m = m\}$. Indeed, obviously, $\Theta_x M \subseteq 1_x M$. If $m \in 1_x M$, then

$$m = 1_x m = \sum_{(x)}(\omega_x\omega_{x^{-1}})m = \sum_{(x)} \omega_x(\omega_{x^{-1}}m) \in \Theta_x M.$$

Analogously, $M\Theta_x = M1_{x^{-1}}$, for each $x \in G$.

Proposition 3.22. *Let $\Theta : G \longrightarrow \mathcal{S}_R(S)$ be a unital partial representation. Then,*

(i) $[\Theta_x] \in \mathbf{PicS}(R)$.
(ii) *If M is an S-bimodule, then*

$$m_l: \begin{array}{ccc} \Theta_x \otimes_R M & \longrightarrow & \Theta_x M \\ u_x \otimes m & \longmapsto & u_x m \end{array} \quad \textit{and} \quad m_r: \begin{array}{ccc} M \otimes_R \Theta_x & \longrightarrow & M\Theta_x \\ m \otimes u_x & \longmapsto & m u_x \end{array}$$

are R-S-bimodule and S-R-bimodule isomorphisms, respectively.
(iii) *Let M be an S-bimodule and let N be an R-subbimodule of M. Then we have the following R-bimodule isomorphisms:*

$$\begin{array}{ccc} \Theta_x \otimes_R N & \longrightarrow & \Theta_x N \\ u_x \otimes n & \longmapsto & u_x n \end{array} \quad \textit{and} \quad \begin{array}{ccc} N \otimes_R \Theta_x & \longrightarrow & N\Theta_x \\ n \otimes u_{x^{-1}} & \longmapsto & n u_{x^{-1}} \end{array}.$$

Proof. (i) Let $\omega_x^i \in \Theta_x$ and $\omega_{x-1}^i \in \Theta_{x^{-1}}$ be such that $\sum_{i=1}^{m} \omega_x^i \omega_{x^{-1}}^i = 1_x$. Define, $f_i : \Theta_x \longrightarrow R$ by $f(u_x) = \omega_{x^{-1}}^i u_x$. Then f_i is right R-linear and

$$\sum_{i=1}^{n} \omega_x^i f_i(u_x) = \sum_{i=1}^{n} \omega_x^i \omega_{x^{-1}}^i u_x = 1_x u_x = u_x,$$

for all $u_x \in \Theta_x$. Consequently, $\{\omega_x^i, f_i\}$ is a dual basis for the right R-module Θ_x. Analogously, let $\overline{\omega}_x^j \in \Theta_x$ and $\overline{\omega}_{x^{-1}}^j \in \Theta_{x^{-1}}$, $j = 1, 2, ..., m$, be such that $\sum_{j=1}^{m} \overline{\omega}_{x^{-1}}^j \overline{\omega}_x^j = 1_{x^{-1}}$, and $g_j : \Theta_x \longrightarrow R$, defined by $g_j(u_x) = u_x \overline{\omega}_{x^{-1}}^j$, then $\{\overline{\omega}_x^j, g_j\}$ is a dual basis for the left R-module Θ_x.

Let $\varphi : \Theta_x \longrightarrow \Theta_x$ be a right R-linear map. Then, $\widetilde{\varphi} = \sum_{i=1}^{n} \varphi(\omega_x^i)\omega_{x^{-1}}^i \in R1_x$ and

$$\widetilde{\varphi} u_x = \sum_{i=1}^{n} \varphi(\omega_x^i)\omega_{x^{-1}}^i u_x = \sum_{i=1}^{n} \varphi(\omega_x^i \omega_{x^{-1}}^i u_x) = \varphi(1_x u_x) = \varphi(u_x),$$

for all $u_x \in \Theta_x$. Then the map $R \longrightarrow \mathrm{End}(\Theta_{x R})$, defined by $r \longmapsto (u_x \mapsto r u_x)$, is surjective. Similarly, the map $R \longrightarrow \mathrm{End}({}_R\Theta_x)$, defined by $r \longmapsto (u_x \mapsto u_x r)$, is surjective. Thus, $[\Theta_x] \in \mathbf{PicS}(R)$.

(ii) Clearly, m_l is a well-defined R-S-bilinear map. Its inverse is given by

$$m_l^{-1}: \begin{array}{ccc} \Theta_x M & \longrightarrow & \Theta_x \otimes_R M, \\ m & \longmapsto & \sum\limits_{(x)} \omega_x \otimes \omega_{x^{-1}} m, \end{array}$$

where $1_x = \sum_{(x)} \omega_x \omega_{x^{-1}}$, with $\omega_x \in \Theta_x$ and $\omega_{x^{-1}} \in \Theta_{x^{-1}}$. Similarly, the inverse of m_r is given by $M\Theta_x \ni m \mapsto \sum_{(x)} m\overline{\omega}_{x^{-1}} \otimes \overline{\omega}_x \in M \otimes \Theta_x$, where $1_{x^{-1}} = \sum_{(\overline{x})} \overline{\omega}_{x^{-1}} \overline{\omega}_x$.

(iii) Note that $\Theta_x N \subseteq M$ is an R-subbimodule of M, such that $1_x n = n$ and $u_{x^{-1}} n \in N$, for all $n \in \Theta_x N$ and $u_{x^{-1}} \in \Theta_{x^{-1}}$. Indeed, if $n \in \Theta_x N$, then $n = \sum_{i=1}^{n} u_x^i n_i$, with $u_x^i \in \Theta_x$ and $n_i \in N$. Since $N \subseteq M$ is an R-subbimodule, we have:

$$u_{x^{-1}} n = \sum_{i=1}^{n} u_{x^{-1}}(u_x^i n_i) = \sum_{i=1}^{n} (\underbrace{u_{x^{-1}} u_x^i}_{\in R1_{x^{-1}}}) n_i \in N.$$

Analogously, $N\Theta_x$ is an R-subbimodule of M which satisfies $n' 1_{x^{-1}} = n$ and $n' u_{x^{-1}} \in N$, for all $n' \in N\Theta_x$ and $u_{x^{-1}} \in \Theta_{x^{-1}}$. Hence, the isomorphism follows as in item (ii). $\square$

Corollary 3.23. *Let $N \in \mathcal{S}_R(S)$. Then $\Theta_x \otimes_R N \simeq \Theta_x N$, for each $x \in G$. In particular, $\Theta_x \otimes_R \Theta_y \simeq \Theta_x \Theta_y$, for all $x, y \in G$.*

Remark 3.24. Let

$$\begin{array}{rccc} \Theta : & G & \longrightarrow & \mathcal{S}_R(S) \\ & x & \longmapsto & \Theta_x \end{array}$$

be a unital partial representation with $\Theta_x \Theta_{x^{-1}} = R1_x$. By Proposition 3.22 and Corollary 3.23 the map

$$\begin{array}{rccc} \underline{\Theta} : & G & \longrightarrow & \mathbf{PicS}(R), \\ & x & \longmapsto & [\Theta_x] \end{array}$$

is a unital partial representation with $\Theta_x \otimes_R \Theta_{x^{-1}} \simeq R1_x$, $(x \in G)$. Let f^Θ be the family of R-bimodule isomorphisms whose members

$$\begin{array}{rccc} f^\Theta_{x,y} : & \Theta_x \otimes_R \Theta_y & \longrightarrow & \Theta_x \Theta_y = 1_x \Theta_{xy}, \\ & u_x \otimes u_y & \longmapsto & u_x u_y \end{array}$$

are induced by the multiplication in S. Then f^Θ is factor set for $\underline{\Theta}$. Therefore, we have a generalized partial crossed product $\Delta(\Theta)$, where each $\Theta_x \subseteq S$ is an R-subbimodule.

Conversely, if $\Theta : G \longrightarrow \mathbf{PicS}(R)$ is a unital partial representation with a factor set f^Θ, then the generalized partial crossed product $\Delta(\Theta)$ is an extension of R with the same unity. In this case, each $\Theta_x \subseteq \Delta(\Theta)$ is an R-subbimodule. By Corollary 3.23, the map $\underline{\Theta} : G \longrightarrow \mathcal{S}_R(\Delta(\Theta))$ with $\underline{\Theta}(x) = \Theta_x$, is a unital partial representation.

Example 3.25. (Partial crossed product) Let $\alpha = (\{D_x\}_{x\in G}, \{\alpha_x\}_{x\in G}, \{\omega_{x,y}\}_{x,y\in G})$ be a unital twisted partial action of G on R with $D_x = R1_x$, for each $x \in G$ (see [18, Definition 2.1]). Consider the partial crossed product $R \rtimes_{\alpha,\omega} G = \bigoplus_{x\in G} D_x \delta_x$, whose multiplication is defined by

$$(u_x \delta_x)(u_y \delta_y) = u_x \alpha_x(u_y 1_{x^{-1}}) \omega_{x,y} \delta_{xy}.$$

By [18, Theorem 2.4], $R \rtimes_{\alpha,\omega} G$ is an associative unital ring. Let

$$\begin{array}{rccc} \Theta : & G & \longrightarrow & \mathcal{S}_R(R \rtimes_{\alpha,\omega} G) \\ & x & \longmapsto & D_x \delta_x \end{array}.$$

It is easy to see that $(D_x \delta_x)(D_{x^{-1}} \delta_{x^{-1}})(D_x \delta_x) \subseteq D_x \delta_x$, for each $x \in G$. On the other hand, if $u_x \delta_x \in D_x \delta_x$, then

$$(u_x\delta_x)(1_{x^{-1}}\delta_{x^{-1}})(\omega^{-1}_{x,x^{-1}}\delta_x) = (u_x\omega_{x,x^{-1}}\delta_1)(\omega^{-1}_{x,x^{-1}}\delta_x) = u_x\omega_{x,x^{-1}}\omega^{-1}_{x,x^{-1}}\delta_x = u_x 1_x\delta_x = u_x\delta_x.$$

Hence, $(D_x\delta_x)(D_{x^{-1}}\delta_{x^{-1}})(D_x\delta_x) = D_x\delta_x$, for each $x \in G$. By [18, Lemma 5.3], we have that

$$(D_x\delta_x)(D_y\delta_y)(D_{y^{-1}}\delta_{y^{-1}}) = (D_{xy}\delta_{xy})(D_{y^{-1}}\delta_{y^{-1}}) \text{ and } (D_x^{-1}\delta_x)(D_x\delta_x)(D_y\delta_y) = (D_{x^{-1}}\delta_{x^{-1}})(D_{xy}\delta_{xy}),$$

for all $x, y \in G$. Consequently, Θ is a partial representation. Observe now that $(D_x\delta_x)(D_{x^{-1}}\delta_{x^{-1}}) = D_x$, for each $x \in G$. Indeed, the inclusion $(D_x\delta_x)(D_{x^{-1}}\delta_{x^{-1}}) \subseteq D_x$ is immediate. Given $u_x \in D_x$, we have $(u_x\omega^{-1}_{x,x^{-1}}\delta_x)(1_{x^{-1}}\delta_x) = u_x\omega^{-1}_{x,x^{-1}}\omega_{x,x^{-1}}\delta_1 = u_x 1_x\delta_1 = u_x\delta_1 \in D_x$. Therefore, Θ is a unital partial representation. By Remark 3.24, we obtain that

$$\begin{array}{rccc} f^{\Theta}_{x,y}: & (D_x\delta_x)\otimes(D_y\delta_y) & \longrightarrow & D_xD_{xy}\delta_{xy} \\ & u_x\delta_x \otimes u_y\delta_y & \longmapsto & u_x\alpha_x(u_y1_{x^{-1}})\omega_{x,y}\delta_{xy} \end{array},$$

is a factor set for Θ. Hence, $\Delta(\Theta) = \bigoplus_{x\in G} D_x\delta_x = R \rtimes_{\alpha,\omega} G$ is a generalized partial crossed product.

4. The group $\mathcal{C}(\Theta/R)$ and partial cohomology

In all what follows the unadorned $\otimes$ will stand for $\otimes_R$, unless otherwise stated.

Let

$$\begin{array}{rccc} \Theta: & G & \longrightarrow & \mathbf{PicS}(R) \\ & x & \longmapsto & [\Theta_x] \end{array}$$

be a unital partial representation with $\varepsilon_x = \Theta_x\otimes\Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$ and let $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x\otimes\Theta_y \longrightarrow 1_x\Theta_{xy},\ x, y \in G\}$ be a factor set for Θ and $\Delta(\Theta)$ the partial generalized crossed product with factor set $f^{\Theta} = \{f^{\Theta}_{x,y} : \Theta_x \otimes \Theta_y \to 1_x\Theta_{xy}, x, y \in G\}$.

Theorem 4.1. *The set*

$$\mathcal{C}(\Theta/R) = \{[\Delta(\Gamma)];\ \Gamma_x|\Theta_x \text{ and } \Gamma_x \otimes \Gamma_{x^{-1}} \simeq R1_x,\ \text{for all } x \in G\}$$

is an abelian group with multiplication defined by

$$[\Delta(\Omega)][\Delta(\Gamma)] = \left[\bigoplus_{x\in G}\Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x\right],$$

where the factor set

$$f^{\Omega\Gamma} = \{f^{\Omega\Gamma}_{x,y} : \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Omega_y \longrightarrow 1_x\Gamma_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Omega_{xy}\}$$

consists of isomorphisms defined in Lemma 3.18.

Proof. By Lemma 3.18 we have $[\Delta(\Omega)][\Delta(\Gamma)] = \left[\bigoplus_{x\in G}\Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x\right] \in \mathcal{C}(\Theta/R)$. First, we need to show that this operation is well defined, i.e. it does not depend on the choice of the representative of the class. Let $[\Delta(\Gamma)] = [\Delta(\Sigma)]$ and $[\Delta(\Omega)] = [\Delta(\Lambda)]$ in $\mathcal{C}(\Theta/R)$, then there is R-bimodule isomorphisms $a_x : \Gamma_x \longrightarrow \Sigma_x$ and $b_x : \Omega_x \longrightarrow \Lambda_x$ and commutative diagrams

$$
\begin{tikzcd}
\Gamma_x \otimes \Gamma_y \arrow[r, "f^{\Gamma}_{x,y}"] \arrow[d, "a_x \otimes a_y"'] & 1_x\Gamma_{xy} \arrow[d, "a_{xy}"] \\
\Sigma_x \otimes \Sigma_y \arrow[r, "f^{\Sigma}_{x,y}"'] & 1_x\Sigma_{xy}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_x \otimes \Omega_y \arrow[r, "f^{\Omega}_{x,y}"] \arrow[d, "b_x \otimes b_y"'] & 1_x\Omega_{xy} \arrow[d, "b_{xy}"] \\
\Lambda_x \otimes \Lambda_y \arrow[r, "f^{\Lambda}_{x,y}"'] & 1_x\Lambda_{xy}
\end{tikzcd}
$$

By definition $[\Delta(\Gamma)][\Delta(\Omega)] = \left[\bigoplus_{x\in G} \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x\right]$ and $[\Delta(\Sigma)][\Delta(\Lambda)] = \left[\bigoplus_{x\in G} \Sigma_x \otimes \Theta_{x^{-1}} \otimes \Lambda_x\right]$. Define the R-bimodule isomorphism $d_{xy} : \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \longrightarrow \Sigma_x \otimes \Theta_{x^{-1}} \otimes \Lambda_x$, by $d_{xy}(v_x \otimes u_{x^{-1}} \otimes \omega_x) = a_x(v_x) \otimes u_{x^{-1}} \otimes b_x(\omega_x)$. We have that the following diagram is commutative

$$
\begin{tikzcd}
\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Omega_y \arrow[r, "f^{\Gamma\Omega}_{x,y}"] \arrow[d, "d_x \otimes d_y"'] & 1_x\Gamma_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Omega_{xy} \arrow[d, "d_{xy}"] \\
\Sigma_x \otimes \Theta_{x^{-1}} \otimes \Lambda_x \otimes \Sigma_y \otimes \Theta_{y^{-1}} \otimes \Lambda_y \arrow[r, "f^{\Sigma\Lambda}_{x,y}"'] & 1_x\Sigma_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Lambda_{xy}
\end{tikzcd}
$$

This implies that $[\Delta(\Gamma)][\Delta(\Omega)] = [\Delta(\Sigma)][\Delta(\Lambda)]$ in $\mathcal{C}(\Theta/R)$ and the assertion follows.

Let us show that with this operation $\mathcal{C}(\Theta/R)$ is, in fact, a group. The associativity is verified analogously to the proof of Lemma 3.18. Given $[\Delta(\Gamma)], [\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$, we have $[\Delta(\Gamma)][\Delta(\Omega)] = \left[\bigoplus_{x\in G} \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x\right]$ and $[\Delta(\Omega)][\Delta(\Gamma)] = \left[\bigoplus_{x\in G} \Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x\right]$. For each $x \in G$, consider the isomorphism λ_x defined by

$$
\begin{tikzcd}
\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \arrow[r] \arrow[rrddd, dashed, bend right, "\lambda_x"'] & \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes R1_{x^{-1}} \arrow[r] & \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \arrow[d] \\
 & & \Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \arrow[d] \\
 & & \Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x \otimes R1_{x^{-1}} \arrow[d] \\
 & & \Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x
\end{tikzcd}
$$

Then it is directly verified that the following diagram is commutative:

$$
\begin{tikzcd}
\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Omega_y \arrow[r, "f^{\Gamma\Omega}_{x,y}"] \arrow[d, "\lambda_x \otimes \lambda_y"'] & 1_x\Gamma_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Omega_{xy} \arrow[d, "1_x\lambda_{xy}"] \\
\Omega_x \otimes \Theta_{x^{-1}} \otimes \Gamma_x \otimes \Omega_y \otimes \Theta_{y^{-1}} \otimes \Gamma_y \arrow[r, "f^{\Omega\Gamma}_{x,y}"'] & 1_x\Omega_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Gamma_{xy}.
\end{tikzcd}
\tag{44}
$$

Thus, $[\Delta(\Gamma)][\Delta(\Omega)] = [\Delta(\Omega)][\Delta(\Gamma)]$ in $\mathcal{C}(\Theta/R)$ and we have the commutativity.

Given $[\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$, we have $[\Delta(\Omega)][\Delta(\Theta)] = \left[\bigoplus_{x\in G} \Omega_x \otimes \Theta_{x^{-1}} \otimes \Theta_x\right]$. For each $x \in G$, consider the R-bimodule isomorphism $\varphi_x : \Omega_x \otimes \Theta_x \otimes \Theta_{x^{-1}} \longrightarrow \Omega_x$ defined by $\varphi_x(\omega_x \otimes u_x \otimes u_{x^{-1}}) = \omega_x(u_x \overset{\Theta}{\circ} u_{x^{-1}})$. Then, the following diagram is commutative

$$\begin{tikzcd}
\Omega_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \otimes \Omega_y \otimes \Theta_{y^{-1}} \otimes \Theta_y \arrow[r, "f^{\Omega\Theta}_{x,y}"] \arrow[d, "\varphi_x \otimes \varphi_y"'] & 1_x\Omega_{xy} \otimes \Theta_{(xy)^{-1}} \otimes \Theta_{xy} \arrow[d, "\varphi_{xy}"] \\
\Omega_x \otimes \Omega_y \arrow[r, "f^{\Omega}_{x,y}"'] & 1_x\Omega_{xy}
\end{tikzcd}$$

Thus, $[\Delta(\Gamma)][\Delta(\Theta)] = [\Delta(\Theta)]$ in $C(\Theta/R)$. Therefore, $[\Delta(\Theta)]$ is the unity in $\mathcal{C}(\Theta/R)$.

Finally, the inverse element of the class $[\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$ is given by $[\Delta(\Omega)]^{-1} = \big[\bigoplus_{x\in G} \Theta_x \otimes \Omega_{x^{-1}} \otimes \Theta_x\big]$. Indeed, by Lemma 3.18 we have $[\Delta(\Omega)]^{-1} \in \mathcal{C}(\Theta/R)$. Moreover, the R-bimodule isomorphism ψ_x : $\Omega_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \otimes \Omega_{x^{-1}} \otimes \Theta_x \to \Theta_x$ defined via

$$\begin{tikzcd}
\Omega_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \otimes \Omega_{x^{-1}} \otimes \Theta_x \arrow[r] \arrow[drr, dashed, "\psi_x"'] & \Omega_x \otimes R1_{x^{-1}} \otimes \Omega_{x^{-1}} \otimes \Theta_x \arrow[r] & \Omega_x \otimes \Omega_{x^{-1}} \otimes \Theta_x \arrow[d] \\
 & & R1_x \otimes \Theta_x \simeq \Theta_x
\end{tikzcd}$$

gives an isomorphism of partial generalized crossed products. Therefore, $\mathcal{C}(\Theta/R)$ is an abelian group. $\square$

Proposition 4.2. *The set*

$$\mathcal{C}_0(\Theta/R) = \{[\Delta(\Gamma)]; \Gamma_x \simeq \Theta_x,\ \textit{for all}\ \ x \in G\},$$

is a subgroup of $\mathcal{C}(\Theta/R)$.

Proof. If $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$, then $\Gamma_x \simeq \Theta_x$, for all $x \in G$. In particular, $\Gamma_x|\Theta_x$ and $\Gamma_x\otimes\Gamma_{x^{-1}} \simeq \Theta_x\otimes\Theta_{x^{-1}} \simeq R1_x$, for all $x \in G$. Then, $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$. Given $[\Delta(\Gamma)], [\Delta(\Omega)] \in \mathcal{C}_0(\Theta/R)$, we have

$$\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \simeq \Theta_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \simeq \Theta_x \text{ and } \Theta_x \otimes \Gamma_{x^{-1}} \otimes \Theta_x \simeq \Theta_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \simeq \Theta_x,$$

for all $x \in G$. Thus, $[\Delta(\Gamma)][\Delta(\Omega)], [\Delta(\Gamma)]^{-1} \in \mathcal{C}_0(\Theta/R)$. Therefore, $\mathcal{C}_0(\Theta/R)$ is a subgroup of $\mathcal{C}(\Theta/R)$. $\square$

Lemma 4.3. *Let* $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$ *and* $a_x : \Gamma_x \longrightarrow \Theta_x$ *be an R-bimodule isomorphism, for each* $x \in G$. *Let* $\tau_{x,y} : 1_x\Theta_{xy} \longrightarrow 1_x\Theta_{xy}$ *be the R-bimodule isomorphism defined by the commutative diagram*

$$\begin{tikzcd}
\Gamma_x \otimes \Gamma_y \arrow[rr, "f^{\Gamma}_{x,y}"] \arrow[d, "a_x \otimes a_y"'] & & 1_x\Gamma_{xy} \arrow[d, "a_{xy}"] \\
\Theta_x \otimes \Theta_y \arrow[r, "f^{\Theta}_{x,y}"'] & 1_x\Theta_{xy} \arrow[r, dashed, "\tau_{x,y}"'] & 1_x\Theta_{xy}
\end{tikzcd}$$

that is, $\tau_{x,y} \circ f^{\Theta}_{x,y} \circ (a_x \otimes a_y) = a_{xy} \circ f^{\Gamma}_{x,y},\ x, y \in G$. *Then,* $\widetilde{\tau}_{-,-}$ *is a normalized element in* $Z^2_{\Theta}(G, \alpha, \mathcal{Z})$, *where* $\widetilde{\tau}_{x,y}$ *is defined in Lemma 3.10.*

Proof. By Lemma 3.10 we have $\widetilde{\tau}_{x,y} \in \mathcal{U}(\mathcal{Z}1_x1_{xy})$ and

$$\widetilde{\tau}_{x,y}(a_x(v_x) \overset{\Theta}{\circ} a_y(v_y)) = a_{xy}(v_x \overset{\Gamma}{\circ} v_y), \tag{45}$$

for all $v_x \in \Gamma_x$ and $v_y \in \Gamma_y$. For $x, y, z \in G$, $v_x \in \Gamma_x, v_y \in \Gamma_y$ and $v_z \in \Gamma_z$, we obtain

$$\begin{aligned}
a_{xyz}((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z) &\overset{(45)}{=} \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}((a_x(v_x) \overset{\Theta}{\circ} a_y(v_y)) \overset{\Theta}{\circ} a_z(v_z)) \\
&= \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}(a_x(v_x) \overset{\Theta}{\circ} (a_y(v_y) \overset{\Theta}{\circ} a_z(v_z))) \\
&\overset{(45)}{=} \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}(a_x(v_x) \overset{\Theta}{\circ} \widetilde{\tau}_{y,z}^{-1} a_{yz}(v_y \overset{\Gamma}{\circ} v_z)) \\
&\overset{(28)}{=} \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})(a_x(v_x) \overset{\Theta}{\circ} a_{yz}(v_y \overset{\Gamma}{\circ} v_z)) \\
&\overset{(45)}{=} \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})\widetilde{\tau}_{x,yz}^{-1} a_{xyz}(v_x \overset{\Gamma}{\circ} (v_y \overset{\Gamma}{\circ} v_z)) \\
&= \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})\widetilde{\tau}_{x,yz}^{-1} a_{xyz}((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z) \\
&= a_{xyz}(\widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})\widetilde{\tau}_{x,yz}^{-1}(v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z).
\end{aligned}$$

Since a_{xyz} is an R-bimodule isomorphism, we see that

$$((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z) = \widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})\widetilde{\tau}_{x,yz}^{-1}((v_x \overset{\Gamma}{\circ} v_y) \overset{\Gamma}{\circ} v_z), \tag{46}$$

for all $v_x \in \Gamma_x, v_y \in \Gamma_y$ and $v_z \in \Gamma_z$. Using the same argument as in Proposition 3.15 we obtain

$$\widetilde{\tau}_{xy,z}\widetilde{\tau}_{x,y}\alpha_x(\widetilde{\tau}_{y,z}^{-1} 1_{x^{-1}})\widetilde{\tau}_{x,yz}^{-1} 1_x = 1_x 1_{xy} 1_{xyz}.$$

This implies that $\widetilde{\tau}_{-,-}^{-1} \in Z^2_\Theta(G, \alpha, \mathcal{Z})$ and consequently $\widetilde{\tau}_{-,-} \in Z^2_\Theta(G, \alpha, \mathcal{Z})$. For $x = 1$, by (36) and (45) we have $\widetilde{\tau}_{1,y}(ra_y(v_y)) = a_y(rv_y)$ for all $r \in R, v_y \in \Gamma_y$. Take $r = 1$. Since a_y is an R-bimodule isomorphism, then $\widetilde{\tau}_{1,y} v_y = v_y$, for all $v_y \in \Gamma_y$. The same argument as above implies that $\widetilde{\tau}_{1,y} = 1_y$. Analogously, $\widetilde{\tau}_{x,1} = 1_x$. Therefore, $\widetilde{\tau}_{-,-}$ is normalized. □

Theorem 4.4. *The map*

$$\begin{array}{rccc} \zeta : & \mathcal{C}_0(\Theta/R) & \longrightarrow & H^2_\Theta(G, \alpha, \mathcal{Z}) \\ & [\Delta(\Gamma)] & \longmapsto & [\widetilde{\tau}_{-,-}], \end{array}$$

where $\widetilde{\tau}_{-,-} \in Z^2(G, \alpha, \mathcal{Z})$ is defined in Lemma 4.3, is a group isomorphism.

Proof. Let $[\Delta(\Gamma)], [\Delta(\Omega)] \in \mathcal{C}_0(\Theta/R)$ and take R-bimodule isomorphisms $a_x : \Gamma_x \longrightarrow \Theta_x$ and $b_x : \Omega_x \longrightarrow \Theta_x$, for $x \in G$. Denote,

$$\tau_{x,y} \circ f^\Theta_{x,y} \circ (a_x \otimes a_y) = a_{xy} \circ f^\Gamma_{x,y}, \text{ and } \gamma_{x,y} \circ f^\Theta_{x,y} \circ (b_x \otimes b_y) = b_{xy} \circ f^\Omega_{x,y}, \quad \forall\, x, y \in G. \tag{47}$$

Let us first show that ζ is well-defined. Suppose that $[\Delta(\Gamma)] = [\Delta(\Omega)]$ in $\mathcal{C}_0(\Theta/R)$. Then there exist R-bimodule isomorphisms $\xi_x : \Gamma_x \longrightarrow \Omega_x$, $x \in G$, such that the following diagram is commutative:

$$\begin{tikzcd} \Gamma_x \otimes \Gamma_y \arrow[r, "f^\Gamma_{x,y}"] \arrow[d, "\xi_x \otimes \xi_y"'] & 1_x\Gamma_{xy} \arrow[d, "\xi_{xy}"] \\ \Omega_x \otimes \Omega_y \arrow[r, "f^\Omega_{x,y}"'] & 1_x\Omega_{xy}. \end{tikzcd} \tag{48}$$

For each $x \in G$ consider the R-bimodule isomorphism $\beta_x : \Theta_x \to \Theta_x$ defined by

$$\beta_x = b_x \circ \xi_x \circ a_x^{-1}, \quad \forall\, x \in G. \tag{49}$$

Claim 4.5. *$\beta_{xy} \circ \tau_{x,y} \circ f^{\Theta}_{x,y} = \gamma_{x,y} \circ f^{\Theta}_{x,y} \circ (\beta_x \otimes \beta_y)$, for all $x, y \in G$.*

Indeed, by (47) and by the commutative diagram (48), we obtain

$$\begin{aligned}
\gamma_{x,y} \circ f^{\Theta}_{x,y} \circ (\beta_x \otimes \beta_y) &= \gamma_{x,y} \circ f^{\Theta}_{x,y} \circ ((b_x \circ \xi_x \circ a_x^{-1}) \otimes (b_y \circ \xi_y \circ a_y^{-1})) \\
&= \gamma_{x,y} \circ f^{\Theta}_{x,y} \circ (b_x \otimes b_y) \circ (\xi_x \otimes \xi_y) \circ (a_x^{-1} \otimes a_y^{-1}) \\
&\overset{(47)}{=} b_{xy} \circ f^{\Omega}_{x,y} \circ (\xi_x \otimes \xi_y) \circ (a_x^{-1} \otimes a_y^{-1}) \\
&= b_{xy} \circ \xi_{xy} \circ f^{\Gamma}_{x,y} \circ (a_x^{-1} \otimes a_y^{-1}) \\
&\overset{(47)}{=} b_{xy} \circ \xi_{xy} \circ a_{xy}^{-1} \circ \tau_{x,y} \circ f^{\Theta}_{x,y} \\
&= \beta_{xy} \circ \tau_{x,y} \circ f^{\Theta}_{x,y}.
\end{aligned}$$

Then, for each $u_x \in \Theta_x$, $u_y \in \Theta_y$ we have $\beta_{xy}(\tau_{x,y}(u_x \overset{\Theta}{\circ} u_y)) = \gamma_{x,y}(\beta_x(u_x) \overset{\Theta}{\circ} \beta_y(u_y))$. On one hand, by (25), we see that $\beta_{xy}(\tau_{x,y}(u_x \overset{\Theta}{\circ} u)) = \beta_{xy}(\widetilde{\tau}_{x,y}(u_x \overset{\Theta}{\circ} u_y)) = \widetilde{\beta}_{xy}\widetilde{\tau}_{x,y}(u_x \overset{\Theta}{\circ} u_y)$. On the other hand, using (25) and (28), we obtain $\gamma_{x,y}(\beta_x(u_x) \overset{\Theta}{\circ} \beta_y(u_y)) = \widetilde{\gamma}_{x,y}(\widetilde{\beta}_x u_x \overset{\Theta}{\circ} \widetilde{\beta}_y u_y) = \widetilde{\gamma}_{x,y}\widetilde{\beta}_x\alpha_x(\widetilde{\beta}_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y)$. Then,

$$\widetilde{\beta}_{xy}\widetilde{\tau}_{x,y}(u_x \overset{\Theta}{\circ} u_y) = \widetilde{\gamma}_{x,y}\widetilde{\beta}_x\alpha_x(\widetilde{\beta}_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y),$$

for all $u_x \in \Theta_x$ and $u_y \in \Theta_y$. Using the argument in Lemma 4.3, we have

$$\widetilde{\beta}_{xy}\widetilde{\tau}_{x,y} = \widetilde{\gamma}_{x,y}\widetilde{\beta}_x\alpha_x(\widetilde{\beta}_y 1_{x^{-1}}), \quad \text{for all} \quad x, y \in G. \tag{50}$$

Let $h : G \longrightarrow \mathcal{Z}$ defined by $h_x = \widetilde{\beta}_x$. Then $h_x \in \mathcal{U}(\mathcal{Z}1_x)$, for all $x \in G$ and we get

$$\widetilde{\tau}_{x,y} = \widetilde{\gamma}_{x,y}\alpha_x(h_y 1_{x^{-1}})h_{xy}^{-1}h_x = \widetilde{\gamma}_{x,y}(\delta^1 h)(x, y), \quad \forall\, x, y \in G.$$

This implies that $[\widetilde{\tau}_{-,-}] = [\widetilde{\gamma}_{-,-}]$ in $H^2_{\Theta}(G, \alpha, \mathcal{Z})$, proving that ζ is well-defined.

To show that ζ is injective, suppose that $\zeta([\Delta(\Gamma)]) = \zeta([\Delta(\Omega)])$ in $H^2_{\Theta}(G, \alpha, \mathcal{Z})$. Then there exist $h : G \longrightarrow \mathcal{Z}$, with $h(x) = h_x \in \mathcal{U}(\mathcal{Z}1_x)$, for all $x \in G$, and

$$\widetilde{\tau}_{x,y} = \widetilde{\gamma}_{x,y}\alpha_x(h_y 1_{x^{-1}})h_{xy}^{-1}h_x, \quad \forall\, x, y \in G.$$

Consider the map

$$\begin{aligned}
\beta_x : \quad \Theta_x &\longrightarrow \Theta_x \\
u_x &\longmapsto h_x u_x.
\end{aligned}$$

Then β_x is an R-bimodule isomorphism and $\widetilde{\beta}_x = h_x$, for all $x \in G$ (see Lemma 3.10). Hence,

$$\widetilde{\tau}_{x,y}\widetilde{\beta}_{xy} = \widetilde{\gamma}_{x,y}\alpha_x(\widetilde{\beta}_y 1_{x^{-1}})\widetilde{\beta}_y, \quad \forall\, x, y \in G. \tag{51}$$

Let $\lambda_x : \Gamma_x \to \Omega_x$, $(x \in G)$, be the isomorphism defined by $\lambda_x = b_x^{-1} \circ \beta_x \circ a_x$. We will verify that following diagram is commutative:

$$\begin{tikzcd}
\Gamma_x \otimes_R \Gamma_y \arrow[r, "f^{\Gamma}_{x,y}"] \arrow[d, "\lambda_x \otimes \lambda_y"'] & 1_x\Gamma_{xy} \arrow[d, "\lambda_{xy}"] \\
\Omega_x \otimes_R \Omega_y \arrow[r, "f^{\Omega}_{x,y}"'] & 1_x\Omega_{xy}.
\end{tikzcd} \tag{52}$$

Firstly, we check that

$$\beta_{xy} \circ \tau_{x,y} \circ f^{\Theta}_{x,y} = \gamma_{x,y} \circ f^{\Theta}_{x,y} \circ (\beta_x \otimes \beta_y), \text{ for all } x, y \in G. \tag{53}$$

Indeed, for $u_x \in \Theta_x$ and $u_y \in \Theta_y$, by (25) we see that $\beta_{xy}(\tau_{x,y}(u_x \overset{\Theta}{\circ} u_y)) = \widetilde{\beta}_{x,y}\widetilde{\tau}_{x,y}(u_x \overset{\Theta}{\circ} u_y)$. Analogously, by (25), (28) and (51) it follows that

$$\gamma_{x,y}(\beta_x(u_x) \overset{\Theta}{\circ} \beta_y(u_y)) = \widetilde{\gamma}_{x,y}(\widetilde{\beta}_x u_x \overset{\Theta}{\circ} \widetilde{\beta}_y u_y) = \widetilde{\gamma}_{x,y}\widetilde{\beta}_x \alpha_x(\widetilde{\beta}_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y) = \widetilde{\beta}_{xy}\widetilde{\tau}_{x,y}(u_x \overset{\Theta}{\circ} u_y)$$

Consequently, (53) holds. Using (47) and (53), we have

$$\begin{aligned}
\lambda_{xy} \circ f^{\Gamma}_{x,y} &= b^{-1}_{xy} \circ \beta_{xy} \circ a_{xy} \circ f^{\Gamma}_{x,y} \overset{(47)}{=} b^{-1}_{xy} \circ \beta_{xy} \circ \tau_{x,y} \circ f^{\Theta}_{x,y} \circ (a_x \otimes a_y) \\
&\overset{(53)}{=} b^{-1}_{xy} \circ \gamma_{xy} \circ f^{\Theta}_{x,y} \circ (\beta_x \otimes \beta_y) \circ (a_x \otimes a_y) \\
&\overset{(47)}{=} b^{-1}_{xy} \circ b_{xy} \circ f^{\Omega}_{x,y} \circ (b^{-1}_x \otimes b^{-1}_y) \circ (\beta_x \otimes \beta_y) \circ (a_x \otimes a_y) \\
&= f^{\Omega}_{x,y} \otimes (\lambda_x \otimes \lambda_y).
\end{aligned}$$

This implies that the diagram (52) is commutative. Thus, we have a partial generalized crossed product isomorphism. Therefore, $[\Delta(\Gamma)] = [\Delta(\Omega)]$ in $\mathcal{C}_0(\Theta/R)$ and ζ is injective.

Let $\sigma : G \times G \longrightarrow \mathcal{Z}$ be a normalized 2-cocycle in $Z^2_{\Theta}(G, \alpha, \mathcal{Z})$. Then

$$\alpha_x(\sigma_{y,z} 1_{x^{-1}})\sigma_{x,yz} = \sigma_{xy,z}\sigma_{x,y}, \ \forall\, x, y, z \in G. \tag{54}$$

Consider the R-bimodule isomorphism $\rho_x : \Theta_x \longrightarrow \Theta_x$ defined by $\rho_x(u_x) = \sigma_{x,x} u_x$, and define $\Sigma_x = \Theta_x$, as R-bimodules. Then, $\rho_x : \Sigma_x \longrightarrow \Theta_x$ is an R-bimodule isomorphism. Let

$$\begin{array}{rccl}
f^{\Sigma}_{x,y} : & \Sigma_x \otimes_R \Sigma_y & \longrightarrow & 1_x \Sigma_{xy} \\
& u_x \otimes u_y & \longmapsto & \sigma_{x,y}(u_x \overset{\Theta}{\circ} u_y).
\end{array}$$

Given $u_x \in \Sigma_x, u_y \in \Sigma_y$ and $u_z \in \Sigma_z$, by (54) we obtain that

$$\begin{aligned}
f^{\Sigma}_{x,yz}(u_x \otimes f^{\Sigma}_{y,z}(u_y \otimes u_z)) &= \sigma_{x,yz}(u_x \overset{\Theta}{\circ} \sigma_{y,z}(u_y \overset{\Theta}{\circ} u_z)) = \sigma_{x,yz}\alpha_x(\sigma_{y,z}1_{x^{-1}})(u_x \overset{\Theta}{\circ} (u_y \overset{\Theta}{\circ} u_z)) \\
&= \sigma_{xy,z}\sigma_{x,y}((u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) = \sigma_{xy,z}(\sigma_{x,y}(u_x \overset{\Theta}{\circ} u_y) \overset{\Theta}{\circ} u_z) \\
&= f^{\Sigma}_{xy,z}(f^{\Sigma}_{x,y}(u_x \otimes u_y) \otimes u_z).
\end{aligned}$$

Thus, $f^{\Sigma} = \{f^{\Sigma}_{x,y} : \sigma_x \otimes \Sigma_y \to 1_x\Sigma_{xy}, \ x, y \in G\}$ is a factor set for Σ and $\Delta(\Sigma) = \bigoplus_x \Sigma_x$ is a partial generalized crossed product with $[\Delta(\Sigma)] \in \mathcal{C}_0(\Theta/R)$. Let $\lambda \in C^1(G, \alpha, \mathcal{Z})$ be defined by $\lambda_x = \sigma^{-1}_{x,x}$. Consider the R-bimodule isomorphism

$$\begin{array}{rccl}
\Upsilon : & 1_x\Theta_{xy} & \to & 1_x\Theta_{xy} \\
& u_x & \mapsto & \sigma_{x,y}(\delta^1\lambda)(x, y)u_{xy}.
\end{array}$$

Then, $\tilde{\Upsilon} = \sigma_{x,y}(\delta^1\lambda)(x, y)$ and

$$\begin{aligned}
\rho_{xy}(f^{\Sigma}_{x,y}(u_x \otimes u_y)) &= \rho_{xy}(\sigma_{x,y}(u_x \overset{\Theta}{\circ} u_y)) = \sigma_{x,y}\rho_{xy}(u_x \overset{\Theta}{\circ} u_y) = \sigma_{x,y}\sigma_{xy,xy}(u_x \overset{\Theta}{\circ} u_y) \\
&= \sigma_{x,y}\sigma_{xy,xy}(\sigma^{-1}_{x,x}\rho_x(u_x) \overset{\Theta}{\circ} \sigma^{-1}_{y,y}\rho_y(u_y)) = \sigma_{x,y}\sigma_{xy,xy}(\sigma^{-1}_{x,x}\rho_x(u_x)\sigma^{-1}_{y,y} \overset{\Theta}{\circ} \rho_y(u_y))
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_{x,y}\sigma_{xy,xy}(\sigma_{x,x}^{-1}\alpha_x(\sigma_{y,y}^{-1}1_{x^{-1}})\rho_x(u_x) \overset{\Theta}{\circ} \rho_y(u_y))\\
&= \sigma_{x,y}\sigma_{xy,xy}\sigma_{x,x}^{-1}\alpha_x(\sigma_{y,y}^{-1}1_{x^{-1}})(\rho_x(u_x) \overset{\Theta}{\circ} \rho_y(u_y))\\
&= \sigma_{x,y}(\delta^1\lambda)(x,y)f_{x,y}^{\Theta}(\rho_x(u_x)\otimes\rho_y(u_y)).
\end{aligned}
$$

This implies that $\rho_{x,y}\circ f_{x,y}^{\Sigma} = \Upsilon\circ f_{x,y}^{\Theta}\circ(\rho_x\otimes\rho_y)$. Thus, $\zeta([\Delta(\Sigma)]) = [\sigma(\delta^1\lambda)] = [\sigma_{-,-}]$ in $H^2_{\Theta}(G,\alpha,\mathcal{Z})$, proving that ζ is onto. Finally, it is easy check that ζ is a group homomorphism. $\square$

5. The seven term exact sequence

In this section we shall construct an exact sequence which generalizes the Miyashita's sequence for non-commutative unital rings [45, Teorema 2.12]. To this end fix a ring extension $R\subseteq S$ with the same unity and a unital partial representation

$$
\begin{array}{rccc}
\Theta: & G & \longrightarrow & \mathcal{S}_R(S),\\
 & x & \longmapsto & \Theta_x,
\end{array}
$$

with $\varepsilon_x = \Theta_x\Theta_{x^{-1}} = R1_x$, for each $x\in G$. Let $\Delta(\Theta)$ be the generalized partial crossed product provided by Remark 3.24 with factor set $f^{\Theta} = \{f_{x,y}^{\Theta} : \Theta_x\otimes\Theta_y \longrightarrow 1_x\Theta_{xy},\ x,y\in G\}$ and the ring and R-bimodule isomorphism $\iota : R\longrightarrow \Theta_1$ as in (35).

5.1. The first exact sequence

In this subsection we establish the initial three term part of the final sequence. Let $\mathcal{P}_{\mathcal{Z}}(S/R)$ be the group defined in Section 2.4. Denote

$$
\mathcal{P}_{\mathcal{Z}}(S/R)^{(G)} = \{\ [P] \xRightarrow{[\phi]} [X]\ \in \mathcal{P}_{\mathcal{Z}}(S/R);\ \Theta_x\phi(P) = \phi(P)\Theta_x \text{ for all } \ x\in G\}.
$$

Remark 5.1. Let $[P] \xRightarrow{[\phi]} [X]\ \in \mathcal{P}_{\mathcal{Z}}(S/R)$. Then,

$$
\Theta_x\phi(P) = \phi(P)\Theta_x \ \text{ if and only if } \ \Theta_x\phi(P)\Theta_{x^{-1}} = \phi(P)1_x, \text{ for each } x\in G. \tag{55}
$$

Indeed, if $\Theta_x\phi(P) = \phi(P)\Theta_x$, for each $x\in G$, then

$$
\Theta_x\phi(P)\Theta_{x^{-1}} = \phi(P)\Theta_x\Theta_{x^{-1}} = \phi(P)1_x, \text{ for each } \ x\in G.
$$

On the other hand, since P is a central $\mathcal{Z}$-bimodule, then $P1_x = 1_xP$, for each $x\in G$. Thus, by the R-bilinearity of ϕ we have $1_x\phi(P) = \phi(P)1_x$, for each $x\in G$. Then

$$
\begin{aligned}
\Theta_x\phi(P)\Theta_{x^{-1}} = \phi(P)1_x &\Rightarrow \Theta_x\phi(P)\Theta_{x^{-1}}\Theta_x = \phi(P)1_x\Theta_x \Rightarrow \Theta_x\phi(P)1_{x^{-1}} = \phi(P)\Theta_x\\
&\Rightarrow \Theta_x 1_{x^{-1}}\phi(P) = \phi(P)\Theta_x \Rightarrow \Theta_x\phi(P) = \phi(P)\Theta_x.
\end{aligned}
$$

Lemma 5.2. *$\mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$ is a subgroup of $\mathcal{P}_{\mathcal{Z}}(S/R)$.*

Proof. It is easy to see that $\mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$ is closed under the multiplication. Let us verify that it is also closed with respect to taking inverses. Let $[P] \xRightarrow{[\phi]} [X]\ \in \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$. As we know from Section 2.4, its inverse in $\mathcal{P}_{\mathcal{Z}}(S/R)$ is given by $[P^*] \xRightarrow{[\phi^*]} [X^*]$. Since ϕ is injective, we may assume $P\subseteq X$ and

$P^* \subseteq X^*$. Observe that $P^* = \{f \in X^*;\ f(P) \subseteq R\}$. It follows that $P^* \cdot 1_x = \{f \in X^*;\ f(P) \subseteq R1_x\}$. Indeed, if $f \in X^*$ is such that $f(P) \subseteq R1_x$, then $f \in P^*$ and

$$(f \cdot 1_x)(p) = f(p)1_x = f(p).$$

Hence, $f \in P^* \cdot 1_x$. On the other hand, if $f \in P^* \cdot 1_x$, then $f = f' \cdot 1_x$, for some $f' \in P^*$. Then,

$$f(p) = (f' \cdot 1_x)(p) = f'(p)1_x \in R1_x, \quad \text{for all} \ \ p \in P,$$

and the claimed equality follows. Let us now see that $\Theta_x \cdot P^* \cdot \Theta_{x^{-1}} = P^* \cdot 1_x$, for each $x \in G$. We have:

$$\begin{aligned}
(\Theta_x \cdot P^* \cdot \Theta_{x^{-1}})(P) &= (P^* \cdot \Theta_{x^{-1}})(P\Theta_x) = (P^* \cdot \Theta_{x^{-1}})(\Theta_x P) \\
&= \Theta_x (P^* \cdot \Theta_{x^{-1}})(P) = \Theta_x [P^*(P)]\Theta_{x^{-1}} \\
&\subseteq \Theta_x R \Theta_{x^{-1}} = R1_x.
\end{aligned}$$

Therefore, $\Theta_x \cdot P^* \cdot \Theta_{x^{-1}} \subseteq P^* \cdot R1_x$, for each $x \in G$. Now, observe that

$$\begin{aligned}
P^* \cdot R1_x = R1_x \cdot P^* \cdot R1_x &= (\Theta_x \Theta_{x^{-1}}) \cdot P^* \cdot (\Theta_x \Theta_{x^{-1}}) \\
&= \Theta_x \cdot (\Theta_{x^{-1}} \cdot P^* \cdot \Theta_x) \cdot \Theta_{x^{-1}} \\
&\subseteq \Theta_x \cdot (P^* \cdot R1_{x^{-1}}) \cdot \Theta_{x^{-1}} \\
&= \Theta_x \cdot P^* \cdot (R1_{x^{-1}} \Theta_{x^{-1}}) \\
&= \Theta_x \cdot P^* \cdot \Theta_{x^{-1}}.
\end{aligned}$$

Consequently, the desired equality follows, and we conclude that $[P^*] \xRightarrow{[\phi^*]} [X^*] \in \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$. $\square$

Proposition 5.3. *The map*

$$\begin{array}{rccc}
\varphi_2: & \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)} & \longrightarrow & \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}, \\
& [P] \xRightarrow{[\phi]} [X] & \longmapsto & [P]
\end{array}$$

is a well-defined group homomorphism.

Proof. Clearly, φ_2 respects the group operations. Thus, it is enough to show that if $[P] \xRightarrow{[\phi]} [X] \in \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$, then $[P] \in \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$. By (55) we have that $\Theta_x \phi(P) \Theta_{x^{-1}} = \phi(PR1_x)$, for each $x \in G$. Since ϕ is R-bilinear, then $\phi(P)$ is an R-subbimodule of X. By Proposition 3.22 and since ϕ is injective, we obtain that f_x is defined by

$$\begin{tikzcd}
\Theta_x \otimes P \otimes \Theta_{x^{-1}} \arrow[r] \arrow[ddrr, dashed, bend right, "f_x"'] & \Theta_x \otimes \phi(P) \otimes \Theta_{x^{-1}} \arrow[r] & \Theta_x \phi(P) \Theta_{x^{-1}} = \phi(P1_x) \arrow[d] \\
& & P1_x \arrow[d] \\
& & P \otimes R1_x
\end{tikzcd}$$

that is,

$$f_x(u_x \otimes p \otimes u_{x^{-1}}) = p' \otimes 1_x,$$

where $\phi(p') = u_x\phi(p)u_{x^{-1}}$ is an R-bimodule isomorphism. Hence $[P] \in \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$. $\square$

Denote

$$\mathbf{Aut}_{R\text{-rings}}(S)^{(G)} = \{f \in \mathbf{Aut}_{R\text{-rings}}(S);\ f(\Theta_x) = \Theta_x,\ \forall\ x \in G\}.$$

Obviously, $\mathbf{Aut}_{R\text{-rings}}(S)^{(G)}$ is a subgroup of $\mathbf{Aut}_{R\text{-rings}}(S)$.

Lemma 5.4. *The following diagram is commutative with exact rows:*

$$\begin{tikzcd}
\mathcal{U}(\mathcal{Z}) \arrow[r, "\mathcal{F}"] & \mathbf{Aut}_{R\text{-}rings}(S) \arrow[r, "\mathcal{E}"] & \mathcal{P}(S/R) \arrow[r, "\vartheta"] & \mathbf{Pic}(R) \\
\mathcal{U}(\mathcal{Z}) \arrow[r] \arrow[u, equal] & \mathbf{Aut}_{R\text{-}rings}(S)^{(G)} \arrow[r, "\varphi_1"] \arrow[u, hook] & \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)} \arrow[r, "\varphi_2"] \arrow[u, hook] & \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \arrow[u, hook]
\end{tikzcd}$$

Proof. The exactness of the first row is given by [27, Proposition 5.1]. The restriction of ϑ to $\mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$ is the homomorphism φ_2 from Proposition 5.3. If $r \in \mathcal{U}(\mathcal{Z})$, then since Θ_x is an R-bimodule, we have that $\mathcal{F}(r) \in \mathbf{Aut}_{R\text{-rings}}(S)^{(G)}$. If $f \in \mathbf{Aut}_{R\text{-rings}}(S)^{(G)}$, then

$$i_f(R) \cdot \Theta_x = Rf(\Theta_x) = R\Theta_x = \Theta_x R = \Theta_x \cdot i_f(R), \text{ for each } \ x \in G.$$

Since R is a central $\mathcal{Z}$-bimodule, then $\mathcal{E}(f) = (\ [R] \xRightarrow{[\iota_f]} [S_f]\ ) \in \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$. Consequently, the homomorphisms of the second row are well-defined. Let us verify that the second row is exact.

The exactness at the first term and the inclusion $\mathcal{E}(\mathbf{Aut}_{R\text{-Rings}}(S)^{(G)}) \subseteq \ker(\varphi_2)$ directly follow from the exactness of the first row. Let $\ [P] \xRightarrow{[\phi]} [X]\ \in \ker(\varphi_2)$. Then there exists $f \in \mathbf{Aut}_{R\text{-rings}}(S)$ such that $\mathcal{E}(f) = (\ [R] \xRightarrow{[\iota_f]} [S_f]\ ) = (\ [P] \xRightarrow{[\phi]} [X]\ )$ in $\mathcal{P}(S/R)$. More specifically, by the proof of [45, Teorema 1.5] we have that if $\lambda : R \longrightarrow P$ is an R-bimodule isomorphism, then, defining α and β by

$$\alpha : S \longrightarrow R \otimes_R S \longrightarrow P \otimes_R S \xrightarrow{\bar{\phi}_r} X \text{ and } \beta : S \longrightarrow S \otimes_R R \longrightarrow S \otimes_R P \xrightarrow{\bar{\phi}_l} X,$$

we may assume that $f = \beta^{-1} \circ \alpha$, and the diagram

$$\begin{tikzcd}
R \arrow[r, "i_f"] \arrow[d, "\lambda"'] & S_f \arrow[d, "\beta"] \\
P \arrow[r, "\phi"'] & X
\end{tikzcd}$$

is commutative. Since $\ [P] \xRightarrow{[\phi]} [X]\ \in \mathcal{P}_{\mathcal{Z}}(S/R)^{(G)}$, then $\phi(P)\Theta_x = \Theta_x\phi(P)$, for each $x \in G$. We shall verify that $f(\Theta_x) = \Theta_x$, for all $x \in G$.

Given $u_x \in \Theta_x$, we have that $\phi(\lambda(1))u_x \in \phi(P)\Theta_x = \Theta_x\phi(P)$, and, consequently, there exist $u_x^i \in \Theta_x$ and $p_i \in P$, $(i = 1, 2, ..., n)$, such that $\phi(\lambda(1))u_x = \sum_{i=1}^{n} u_x^i \phi(p_i)$. As λ is an isomorphism, there exists $r_i \in R$ such that $\lambda(r_i) = p_i$, for all $i = 1, 2, ..., n$. Hence,

$$\alpha(u_x) = \phi(\lambda(1))u_x = \sum_{i=1}^{n} u_x^i \phi(p_i) = \sum_{i=1}^{n} u_x^i \phi(\lambda(r_i)) = \sum_{i=1}^{n} u_x^i \beta(i_f(r_i)) = \sum_{i=1}^{n} u_x^i \beta(r_i) = \sum_{i=1}^{n} \beta(u_x^i r_i).$$

Thus,

$$f(u_x) = (\beta^{-1} \circ \alpha)(u_x) = \sum_{i=1}^{n} u_x^i r_i \in \Theta_x.$$

Consequently, $f(\Theta_x) \subseteq \Theta_x$, for all $x \in G$.

On the other hand, given $u_x \in \Theta_x$, we have $u_x\phi(\lambda(1)) \in \Theta_x\phi(P)$, and thus there exist $v_x^j \in \Theta_x$ and $p_j \in P$ with $j = 1, ..., m$, such that $u_x\phi(\lambda(1)) = \sum_{j=1}^{m} \phi(p_j)v_x^j$. Using again that λ is an isomorphism, we get that there exist elements $r_j \in R$ such that $\lambda(r_j) = p_j$ for $j = 1, ..., m$. Then

$$\beta(u_x) = u_x\phi(\lambda(1)) = \sum_{j=1}^{m} \phi(p_j)v_x^j = \sum_{j=1}^{m} \phi(\lambda(r_j))v_x^j = \sum_{j=1}^{m} \phi(\lambda(1))r_j v_x^j = \alpha\left(\sum_{j=1}^{m} r_j v_x^j\right).$$

Therefore,

$$u_x = (\beta^{-1} \circ \alpha)\left(\sum_{j=1}^{m} r_j v_x^j\right) = f\left(\sum_{j=1}^{m} r_j v_x^j\right) \in f(\Theta_x).$$

Hence, $\Theta_x \subseteq f(\Theta_x)$ and, consequently, we obtain the desired equality. It follows that $f \in \mathbf{Aut}_{R\text{-rings}}(S)^{(G)}$ and, thus, $[P] \overset{[\phi]}{\Longrightarrow} [X] \in \mathcal{E}(\mathbf{Aut}_{R\text{-rings}}(S)^{(G)})$. $\square$

Since $R \subseteq \Delta(\Theta)$ is a ring extension with the same unity, Lemma 5.4 is applicable:

Theorem 5.5. *The following sequence of groups and homomorphisms is exact:*

$$1 \longrightarrow H^1_\Theta(G, \alpha, \mathcal{Z}) \xrightarrow{\varphi_1} \mathcal{P}_\mathcal{Z}(\Delta(\Theta)/R)^{(G)} \xrightarrow{\varphi_2} \mathbf{Pic}_\mathcal{Z}(R) \cap \mathbf{PicS}_\mathcal{Z}(R)^{\alpha^*}.$$

Proof. By Lemma 5.4, the sequence

$$\mathcal{U}(\mathcal{Z}) \xrightarrow{\mathcal{F}} \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)} \longrightarrow \mathcal{P}_\mathcal{Z}(\Delta(\Theta)/R)^{(G)} \xrightarrow{\varphi_2} \mathbf{Pic}_\mathcal{Z}(R) \cap \mathbf{PicS}_\mathcal{Z}(R)^{\alpha^*}$$

is exact. Hence,

$$1 \longrightarrow \frac{\mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}}{\mathrm{Im}(\mathcal{F})} \longrightarrow \mathcal{P}_\mathcal{Z}(\Delta(\Theta)/R)^{(G)} \xrightarrow{\varphi_2} \mathbf{Pic}_\mathcal{Z}(R) \cap \mathbf{PicS}_\mathcal{Z}(R)^{\alpha^*}$$

is exact. Thus, it suffices to show that there exists a group isomorphism $\dfrac{\mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}}{\mathrm{Im}(\mathcal{F})} \simeq H^1_\Theta(G, \alpha, \mathcal{Z})$.

Let $f \in \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}$. Since $f(\Theta_x) = \Theta_x$ for all $x \in G$ and f fixes each element of R, restricting f to Θ_x, we get an R-bimodule isomorphism $f_x : \Theta_x \longrightarrow \Theta_x$, for each $x \in G$. Let $\widetilde{f_x} \in \mathcal{U}(\mathcal{Z}1_x)$ be as in

Lemma 3.10. As f is a ring automorphism, we have that $f(u_x \overset{\Theta}{\circ} u_y) = f(u_x) \overset{\Theta}{\circ} f(u_y)$, for all $u_x \in \Theta_x$ and $u_y \in \Theta_y$. Then

$$f_{xy}(u_x \overset{\Theta}{\circ} u_y) = f_x(u_x) \overset{\Theta}{\circ} f_y(u_y),\ u_x \in \Theta_x, u_y \in \Theta_y.$$

By (25) we obtain

$$\begin{gathered} f_{xy}(u_x \overset{\Theta}{\circ} u_y) = \widetilde{f_{xy}}(u_x \overset{\Theta}{\circ} u_y), \\ f_x(u_x) \overset{\Theta}{\circ} f_y(u_y) = \widetilde{f}_x u_x \overset{\Theta}{\circ} \widetilde{f}_y u_y = \widetilde{f}_x u_x \widetilde{f}_y \overset{\Theta}{\circ} u_y \overset{(28)}{=} \widetilde{f}_x \alpha_x(\widetilde{f}_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y). \end{gathered}$$

Consequently, $\widetilde{f_{xy}}(u_x \overset{\Theta}{\circ} u_y) = \widetilde{f}_x \alpha_x(\widetilde{f}_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y)$, for all $u_x \in \Theta_x$ and $u_y \in \Theta_y$. Using the same argument as in Proposition 3.15 we obtain

$$\widetilde{f}_x \alpha_x(\widetilde{f}_y 1_{x^{-1}}) = \widetilde{f_{xy}} 1_x,\ \text{for all } x, y \in G.$$

Then

$$\begin{array}{rccc} \widetilde{f}: & G & \longrightarrow & \mathcal{Z}, \\ & x & \longmapsto & \widetilde{f}_x, \end{array}$$

belongs to $Z^1_\Theta(G, \alpha, \mathcal{Z})$. Since $\widetilde{f}_1 = f_1(1) = 1$, it follows that $\widetilde{f}$ is a normalized element in $Z^1_\Theta(G, \alpha, \mathcal{Z})$. We have the following homomorphism of groups:

$$\begin{array}{rccc} \Psi: & \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)} & \longrightarrow & Z^1_\Theta(G, \alpha, \mathcal{Z}) \\ & f & \longmapsto & \widetilde{f} \end{array}.$$

Conversely, if $\sigma \in Z^1_\Theta(G, \alpha, \mathcal{Z})$ is a normalized 1-cocycle, then

$$\alpha_x(\sigma_y 1_{x^{-1}})\sigma_x = \sigma_{xy} 1_x, \quad \text{for all } x, y \in G. \tag{56}$$

For each $x \in G$ define

$$\begin{array}{rccc} g_x: & \Theta_x & \longrightarrow & \Theta_x, \\ & u_x & \longmapsto & \sigma_x u_x. \end{array}$$

Since $\sigma_x \in \mathcal{U}(\mathcal{Z}1_x)$, it follows by Lemma 3.10 that g_x is an R-bilinear isomorphism and $\widetilde{g_x} = \sigma_x$, for each $x \in G$. Consider the map

$$\begin{array}{rccc} g := \bigoplus_{x \in G} g_x: & \Delta(\Theta) & \longrightarrow & \Delta(\Theta) \\ & u_x & \longrightarrow & g_x(u_x) \end{array}.$$

Then $g_x(\Theta_x) = \Theta_x$, for all $x \in G$. Since σ is normalized, we have:

$$g(r) = g_1(r) = \sigma_1 r = 1r = r, \quad \forall\, r \in R.$$

Hence, g fixes each element of R. Now, given $u_x \in \Theta_x$ and $u_y \in \Theta_y$, we compute:

$$\begin{aligned} g(u_x \overset{\Theta}{\circ} u_y) &= \sigma_{xy}(u_x \overset{\Theta}{\circ} u_y) \overset{(56)}{=} \sigma_x \alpha_x(\sigma_y 1_{x^{-1}})(u_x \overset{\Theta}{\circ} u_y) = (\sigma_x \alpha_x(\sigma_y 1_{x^{-1}}) u_x \overset{\Theta}{\circ} u_y) \\ &\overset{(28)}{=} (\sigma_x u_x \sigma_y \overset{\Theta}{\circ} u_y) = (\sigma_x u_x \overset{\Theta}{\circ} \sigma_y u_y) = (g_x(u_x) \overset{\Theta}{\circ} g_y(u_y)). \end{aligned}$$

Therefore, $g \in \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}$ and we have the map

$$\begin{array}{rccl} \Phi: & Z^1_\Theta(G,\alpha,\mathcal{Z}) & \longrightarrow & \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta)/R)^{(G)}, \\ & \sigma & \longmapsto & g. \end{array}$$

Let us check that Φ is the inverse of Ψ. If $f \in \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}$, then

$$\Phi(\Psi(f)) = \Phi(\widetilde{f}) = \bigoplus_{x\in G} g_x,$$

where $g_x(u_x) = \widetilde{f}_x u_x = f_x(u_x)$, for all $u_x \in \Theta_x$, $x \in G$. Hence, $\Psi(\Phi(f)) = f$, for all $f \in \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}$.

On the other hand, if $\sigma \in Z^1_\Theta(G,\alpha,\mathcal{Z})$, then

$$\Psi(\Phi(\sigma)) = \Psi(g) = \widetilde{g},$$

where $g_x(u_x) = \sigma_x u_x$, for each $x \in G$. Thus, $\widetilde{g_x} = \sigma_x$, for all $x \in G$. Consequently, $\Psi(\Phi(\sigma)) = \sigma$, for all $\sigma \in Z^1_\Theta(G,\alpha,\mathcal{Z})$. Therefore, $\Psi : \mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)} \longrightarrow Z^1_\Theta(G,\alpha,\mathcal{Z})$ is a group isomorphism. Now, if $r \in \mathcal{U}(\mathcal{Z})$, then

$$\mathcal{F}(r)(u_x) = r u_x r^{-1} = r\alpha_x(r^{-1}1_{x^{-1}})u_x = (\delta^0 r^{-1})(x)u_x, \text{ for all } u_x \in \Theta_x.$$

Hence, $\Psi(\mathcal{F}(r)) = \delta^0 r^{-1} \in B^1_\Theta(G,\alpha,\mathcal{Z})$. Conversely, if $\sigma \in B^1_\Theta(G,\alpha,\mathcal{Z})$, then there exists $r \in \mathcal{U}(\mathcal{Z})$ such that $\delta^0 r = \sigma$, that is,

$$\sigma(x) = \alpha_x(r1_{x^{-1}})r^{-1}.$$

Then, $\mathcal{F}(r^{-1})(u_x) = r^{-1}u_x r = r^{-1}\alpha_x(r1_{x^{-1}})u_x$, for all $u_x \in \Theta_x$. Thus, $\widetilde{\mathcal{F}(r)}_x = \sigma(x)$, for each $x \in G$. Consequently, $\Psi(\mathcal{F}(r)) = \sigma$. Thus $\Psi(\mathrm{Im}(\mathcal{F})) = B^1_\Theta(G,\alpha,\mathcal{Z})$ and we have the isomorphism

$$\frac{\mathbf{Aut}_{R\text{-rings}}(\Delta(\Theta))^{(G)}}{\mathrm{Im}(\mathcal{F})} \simeq H^1_\Theta(G,\alpha,\mathcal{Z}),$$

as desired. $\square$

5.2. The second exact sequence

Let

$$\mathbf{Pic}_\mathcal{Z}(R)^{(G)} = \{[P] \in \mathbf{Pic}_\mathcal{Z}(R); P \otimes \Theta_x \otimes P^{-1}|\Theta_x, \text{ for all } x \in G\}.$$

Lemma 5.6. *$\mathbf{Pic}_\mathcal{Z}(R)^{(G)}$ is a subgroup of $\mathbf{Pic}_\mathcal{Z}(R)$ which contains $\mathbf{Pic}_\mathcal{Z}(R) \cap \mathbf{PicS}_\mathcal{Z}(R)^{\alpha^*}$.*

Proof. Let $[P], [Q] \in \mathbf{Pic}_\mathcal{Z}(R)^{(G)}$. Since $Q \otimes \Theta_x \otimes Q^{-1}|\Theta_x$, it follows by compatibility with the tensor product that $P \otimes Q \otimes \Theta_x \otimes Q^{-1} \otimes P^{-1}|P \otimes \Theta_x \otimes P^{-1}$. Therefore, by transitivity, we obtain that

$$P \otimes Q \otimes \Theta_x \otimes Q^{-1} \otimes P^{-1}|\Theta_x, \quad \text{for all } x \in G.$$

Hence, $[P \otimes Q] \in \mathbf{Pic}_\mathcal{Z}(R)^{(G)}$. Let us verify that $\mathbf{Pic}_\mathcal{Z}(R)^{(G)}$ is also closed with respect to inverses. Take $[P] \in \mathbf{Pic}_\mathcal{Z}(R)^{(G)}$. Then

$$\begin{aligned}P^{-1}\otimes\Theta_x\otimes P &\simeq P^{-1}\otimes\Theta_x\otimes R1_{x^{-1}}\otimes P\simeq P^{-1}\otimes\Theta_x\otimes P\otimes R1_{x^{-1}}\\ &\simeq P^{-1}\otimes\Theta_x\otimes P\otimes\Theta_{x^{-1}}\otimes\Theta_x\simeq P^{-1}\otimes\Theta_x\otimes P\otimes\Theta_{x^{-1}}\otimes R\otimes\Theta_x\\ &\simeq P^{-1}\otimes\Theta_x\otimes P\otimes\Theta_{x^{-1}}\otimes P^{-1}\otimes P\otimes\Theta_x.\end{aligned}$$

Thus, since $P\otimes\Theta_x\otimes P^{-1}|\Theta_x$, for each $x\in G$, we have

$$P^{-1}\otimes\Theta_x\otimes P\otimes\Theta_{x^{-1}}\otimes P^{-1}\otimes P\otimes\Theta_x|P^{-1}\otimes\Theta_x\otimes\Theta_{x^{-1}}\otimes P\otimes\Theta_x.$$

But,

$$P^{-1}\otimes\Theta_x\otimes\Theta_{x^{-1}}\otimes P\otimes\Theta_x\simeq P^{-1}\otimes R1_x\otimes P\otimes\Theta_x\simeq P^{-1}\otimes P\otimes R1_x\otimes\Theta_x\simeq\Theta_x.$$

Consequently, $P^{-1}\otimes\Theta_x\otimes P|\Theta_x$, for each $x\in G$, showing that $[P^{-1}]\in\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$. Therefore, $\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ is a subgroup of $\mathbf{Pic}_{\mathcal{Z}}(R)$.

Let $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)\cap\mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$. By (30) we have that $\Theta_x\otimes P\simeq P\otimes\Theta_x$, for each $x\in G$. As $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)$, it follows that

$$P\otimes\Theta_x\otimes P^{-1}\simeq\Theta_x,\ \forall\, x\in G,\ [P]\in\mathbf{Pic}_{\mathcal{Z}}(R)\cap\mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}. \tag{57}$$

In particular, $P\otimes\Theta_x\otimes P^{-1}|\Theta_x$, for all $x\in G$. Hence, $\mathbf{Pic}_{\mathcal{Z}}(R)\cap\mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}\subseteq\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$. □

Our next purpose is to construct a generalized partial crossed product starting with an element $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$. We begin by producing a unital partial representation.

Lemma 5.7. *Let $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ and denote $\Omega^P_x=P\otimes\Theta_x\otimes P^{^{-1}}$, $(x\in G)$. Then*

$$\begin{array}{cccc}\Omega^P: & G & \longrightarrow & \mathbf{PicS}(R),\\ & x & \longmapsto & [\Omega^P_x]\end{array}$$

is a unital partial representation with $\Omega^P_x\otimes\Omega^P_{x^{-1}}\simeq R1_x$ and $\Omega^P_x|\Theta_x$, for all $x\in G$.

Proof. Obviously, $[\Omega^P_1]=[R]$ and $\Omega^P_x|\Theta_x$, for all $x\in G$. Given $x,y\in G$, we have:

$$\begin{aligned}\Omega^P_x\otimes\Omega^P_y\otimes\Omega^P_{y^{-1}}&=P\otimes\Theta_x\otimes P^{-1}\otimes P\otimes\Theta_y\otimes P^{-1}\otimes P\otimes\Theta_{y^{-1}}\otimes P^{-1}\\ &\simeq P\otimes\Theta_x\otimes\Theta_y\otimes\Theta_{y^{-1}}\otimes P^{-1}\simeq P\otimes\Theta_{xy}\otimes\Theta_{y^{-1}}\otimes P^{-1}\\ &\simeq P\otimes\Theta_{xy}\otimes P^{-1}\otimes P\otimes\Theta_{y^{-1}}\otimes P^{-1}=\Omega^P_{xy}\otimes\Omega^P_{y^{-1}}.\end{aligned}$$

Similarly, we obtain that $\Omega^P_{x^{-1}}\otimes\Omega^P_x\otimes\Omega^P_y\simeq\Omega^P_{x^{-1}}\otimes\Omega^P_{xy}$, for all $x,y\in G$. Moreover,

$$\begin{aligned}\Omega^P_x\otimes\Omega^P_{x^{-1}}&=P\otimes\Theta_x\otimes P^{^{-1}}\otimes P\otimes\Theta_{x^{-1}}\otimes P^{^{-1}}\simeq P\otimes\Theta_x\otimes\Theta_{x^{-1}}\otimes P^{^{-1}}\\ &\simeq P\otimes R1_x\otimes P^{^{-1}}\simeq R1_x\otimes P\otimes P^{-1}\simeq R1_x\otimes R\simeq R1_x,\quad\forall\, x\in G.\end{aligned}$$

Therefore, Ω^P is a unital partial representation with $\Omega^P_x\otimes\Omega^P_{x^{-1}}\simeq R1_x$, for all $x\in G$. □

Given $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ let $P^{-1}\otimes P\xrightarrow{\mathfrak{r}}R\xleftarrow{\mathfrak{l}}P\otimes P^{-1}$ be R-bimodule isomorphisms. Define the R-bimodule isomorphisms $f^P_{x,y}:\Omega^P_x\otimes\Omega^P_y\longrightarrow 1_x\Omega^P_{xy}$ via

$$
\begin{tikzcd}
P\otimes\Theta_x\otimes P^{-1}\otimes P\otimes\Theta_y\otimes P^{-1} \arrow[r] \arrow[rrd, dashed, "f^P_{x,y}"'] & P\otimes\Theta_x\otimes\Theta_y\otimes P^{-1} \arrow[r] & P\otimes 1_x\Theta_{xy}\otimes P^{-1} \arrow[d] \\
 & & 1_xP\otimes\Theta_{xy}\otimes P^{-1},
\end{tikzcd}
$$

where the first isomorphism is induced by $\mathfrak{r}$, that is,

$$f^P_{x,y}(p_1\otimes u_x\otimes \bar{p}_1\otimes p_2\otimes u_y\otimes \bar{p}_2) = p_1\otimes (u_x\mathfrak{r}(\bar{p}_1\otimes p_2)\overset{\Theta}{\circ} u_y)\otimes \bar{p}_2,$$

for $u_x\in\Theta_x, u_y\in\Theta_y$, $p_1,p_2\in P$ and $\bar{p}_1,\bar{p}_2\in P^{-1}$. Given $x,y,z\in G$, it can be directly seen that the following diagram is commutative

$$
\begin{tikzcd}
P\otimes\Theta_x\otimes P^{-1}\otimes P\otimes\Theta_y\otimes P^{-1}\otimes P\otimes\Theta_z\otimes P^{-1} \arrow[r] \arrow[d] & 1_xP\otimes\Theta_{xy}\otimes P^{-1}\otimes P\otimes\Theta_z\otimes P^{-1} \arrow[d] \\
P\otimes\Theta_x\otimes P^{-1}\otimes 1_yP\otimes\Theta_{yz}\otimes P^{-1} \arrow[r] & 1_x1_{xy}P\otimes\Theta_{xyz}\otimes P^{-1}.
\end{tikzcd}
$$

Consequently, $f^P_{xy,z}\circ(f^P_{x,y}\otimes\Omega^P_z) = f^P_{x,yz}\circ(\Omega^P_x\otimes f^P_{y,z})$. Therefore, $f^P = \{f^P_{x,y} : \Omega^P_x\otimes\Omega^P_y \longrightarrow 1_x\Omega^P_{xy},\ x,y\in G\}$ is a factor set for Ω^P and we have a generalized partial crossed product

$$\Delta(\Omega^P) = \bigoplus_{x\in G} P\otimes\Theta_x\otimes P^{-1},$$

such that $P\otimes\Theta_x\otimes P^{-1}|\Theta_x$ and $\Omega^P_x\otimes\Omega^P_{x^{-1}}\simeq R1_x$, for each $x\in G$. Hence, $[\Delta(\Omega^P)]\in\mathcal{C}(\Theta/R)$, for all $[P]\in\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$.

Remark 5.8. It is important to observe that the isomorphism class of $\Delta(\Omega^P)$ in $\mathcal{C}(\Theta/R)$ does not depend on the choice of the isomorphism $\mathfrak{r}$ used to define the isomorphism $f^P_{x,y}$ in Ω^P.

With respect to the generalized partial crossed product $\Delta(\Omega^P)$ we know that there exists an R-bimodule and a ring isomorphism $\nu_P : R\longrightarrow P\otimes\Theta_1\otimes P^{-1}$ with commutative diagrams like in (35). We shall specify this isomorphism in the next remark, which will be used later.

Remark 5.9. Let $\iota : R\longrightarrow\Theta_1$ be the R-bimodule and ring isomorphism which satisfies the commutative diagrams in (35). Let $\nu_P : R\longrightarrow P\otimes\Theta_1\otimes P^{-1}$ be the isomorphism defined by

$$\nu_P(r) = \sum_{k=1}^{n} rp_k\otimes\iota(1)\otimes\bar{p}_k,$$

where $\sum_{k=1}^{n}\mathfrak{l}(p_k\otimes\bar{p}_k) = 1$. Then, ν_P satisfies the commutative diagrams

$$
\begin{tikzcd}
R\otimes\Omega^P_x \arrow[rr, "\simeq"] \arrow[dr, "\nu_P"'] & & \Omega^P_x \\
 & \Omega^P_1\otimes\Omega^P_x \arrow[ur, "f^P_{1,x}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^P_x\otimes R \arrow[rr, "\simeq"] \arrow[dr, "\nu_P"'] & & \Omega^P_x \\
 & \Omega^P_x\otimes\Omega^P_1 \arrow[ur, "f^P_{x,1}"'] &
\end{tikzcd}
$$

Indeed, given $r\in R$, $p\in P, u_x\in\Theta_x$ and $\bar{p}\in P^{-1}$, we have

$$
\begin{aligned}
f^P_{1,x}(\nu_P(r \otimes p \otimes u_x \otimes \bar{p})) &= \sum_{k=1}^{n} f^P_{1,x}(rp_k \otimes \iota(1) \otimes \bar{p}_k \otimes p \otimes u_x \otimes \bar{p}) = \sum_{k=1}^{n} rp_k \otimes (\iota(1) \overset{\Theta}{\circ} \mathfrak{r}(\bar{p}_k \otimes p)u_x) \otimes \bar{p} \\
&= \sum_{k=1}^{n} rp_k \otimes \mathfrak{r}(\bar{p}_k \otimes p)u_x \otimes \bar{p} = \sum_{k=1}^{n} rp_k\mathfrak{r}(\bar{p}_k \otimes p) \otimes u_x \otimes \bar{p} \\
&= \sum_{k=1}^{n} r\mathfrak{l}(p_k \otimes \bar{p}_k)p \otimes u_x \otimes \bar{p} = rp \otimes u_x \otimes \bar{p}.
\end{aligned}
$$

Analogously, we obtain the commutativity of the second diagram.

Theorem 5.10. *The map*

$$
\begin{array}{rccc}
\mathcal{L}: & \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} & \longrightarrow & \mathcal{C}(\Theta/R), \\
& [P] & \longmapsto & [\Delta(\Omega^P)]
\end{array}
$$

is a group homomorphism.

Proof. In order to see that $\mathcal{L}$ is well-defined, consider $[P] = [Q]$ in $\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$. Then there exist R-bimodule isomorphisms $\varphi : Q \longrightarrow P$ and $\overline{\varphi} : Q^{-1} \longrightarrow P^{-1}$. Let $\mathfrak{r} : P^{-1} \otimes P \longrightarrow R$ be an R-bimodule isomorphism and let $\mathfrak{r}' : Q^{-1} \otimes Q \longrightarrow R$ be the R-bimodule isomorphism defined by $\mathfrak{r}'(\bar{q} \otimes q) = \mathfrak{r}(\overline{\varphi}(\bar{q}) \otimes \varphi(q))$. For each $x \in G$ it is easy to see that the R-bimodule isomorphisms

$$
\begin{array}{rccc}
\varphi_x: & Q \otimes \Theta_x \otimes Q^{-1} & \longrightarrow & P \otimes \Theta_x \otimes P^{-1}, \\
& q \otimes u_x \otimes \bar{q} & \longmapsto & \varphi(q) \otimes u_x \otimes \overline{\varphi}(\bar{q}),
\end{array}
$$

gives a isomorphism of partial generalized crossed products. Thus, $[\Delta(\Omega^P)] = [\Delta(\Omega^Q)]$ in $\mathcal{C}(\Theta/R)$.

Let $[P], [Q] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ and let, furthermore,

$$
P \otimes P^{-1} \xrightarrow{\mathfrak{l}} R \xleftarrow{\mathfrak{r}} P^{-1} \otimes P \quad \text{and} \quad Q \otimes Q^{-1} \xrightarrow{\mathfrak{l}'} R \xleftarrow{\mathfrak{r}'} Q^{-1} \otimes Q
$$

be R-bimodule isomorphisms. Then,

$$
\mathcal{L}([P \otimes Q]) = \left[\bigoplus_{x \in G} P \otimes Q \otimes \Theta_x \otimes Q^{-1} \otimes P^{-1} \right].
$$

On the other hand,

$$
\mathcal{L}([P])\mathcal{L}([Q]) = \left[\bigoplus_{x \in G} P \otimes \Theta_x \otimes P^{-1} \otimes \Theta_{x^{-1}} \otimes Q \otimes \Theta_x \otimes Q^{-1} \right].
$$

Since $Q \otimes \Theta_x \otimes Q^{-1} | \Theta_x$, then $\Theta_{x^{-1}} \otimes Q \otimes \Theta_x \otimes Q^{-1} | R$. As P^{-1} is a central $\mathcal{Z}$-bimodule, by Corollary 2.12 we have an R-bimodule isomorphism $P^{-1} \otimes \Theta_{x^{-1}} \otimes Q \otimes \Theta_x \otimes Q^{-1} \simeq \Theta_{x^{-1}} \otimes Q \otimes \Theta_x \otimes Q^{-1} \otimes P^{-1}$. Similarly, since Q is a central $\mathcal{Z}$-bimodule and $R1_x | R$, we have an R-bimodule isomorphism $R1_x \otimes Q \longrightarrow Q \otimes R1_x$, for each $x \in G$. Define the R-bimodule isomorphism $F_x : \Omega^P_x \otimes \Theta_{x^{-1}} \otimes \Omega^Q_x \longrightarrow \Omega^{P \otimes Q}_x$ by:

$$\begin{tikzcd}
P\otimes\Theta_x\otimes P^{-1}\otimes\Theta_{x^{-1}}\otimes Q\otimes\Theta_x\otimes Q^{-1} \arrow[r] \arrow[dddr, dashed, bend right, "F_x"'] & P\otimes\Theta_x\otimes\Theta_{x^{-1}}\otimes Q\otimes\Theta_x\otimes Q^{-1}\otimes P^{-1} \arrow[d] \\
& P\otimes R1_x\otimes Q\otimes\Theta_x\otimes Q^{-1}\otimes P^{-1} \arrow[d] \\
& P\otimes Q\otimes R1_x\otimes\Theta_x\otimes Q^{-1}\otimes P^{-1} \arrow[d] \\
& P\otimes Q\otimes\Theta_x\otimes Q^{-1}\otimes P^{-1}.
\end{tikzcd}$$

Then routine arguments show that $F = \{F_x : \Omega_x^P \otimes \Theta_{x^{-1}} \otimes \Omega_x^Q \longrightarrow \Omega_x^{P\otimes Q},\ x \in G\}$ is an isomorphism of partial generalized crossed products. Therefore, $\mathcal{L}$ is a group homomorphism. $\square$

By (57), if $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$, then $[\Delta(\Omega^P)] \in \mathcal{C}_0(\Theta/R)$. We define the homomorphism $\varphi_3 : \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \longrightarrow \mathcal{C}_0(\Theta/R)$ as a restriction of $\mathcal{L}$, that is,

$$\begin{array}{rccc}\varphi_3 : & \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} & \longrightarrow & \mathcal{C}_0(\Theta/R) \\ & [P] & \longmapsto & [\Delta(\Omega^P)] \end{array}.$$

For reader's convenience we separate in the next lemma a fact observed in [26, p. 161].

Lemma 5.11. *Let $R \subseteq S$ be a ring extension with the same unity and let P be an R-bimodule. Denote by μ the multiplication in S.*

(i) *Let $\rho : S \otimes_R P \longrightarrow P \otimes_R S$ be an R-bilinear map such that*

$$(P \otimes \mu) \circ (\rho \otimes S) \circ (S \otimes \rho) = \rho \circ (\mu \otimes P), \tag{58}$$

$$\rho(1 \otimes p) = p \otimes 1, \quad \textit{for all } p \in P. \tag{59}$$

If M is a unital left S-module, then $P \otimes_R M$ possesses a structure of a left S-module, given by

$$s * (p \otimes m) = \sum_{i=1}^{n} p_i \otimes s_i m,$$

where $\rho(s \otimes p) = \sum_{i=1}^{n} p_i \otimes s_i$. Moreover, if M is a unital S-bimodule, then $P \otimes_R M$ is a unital S-bimodule, where the structure of a right S-module is given by the structure of M.

(ii) *Let $\rho' : P \otimes_R S \longrightarrow S \otimes_R P$ be an R-bilinear map such that*

$$(\mu \otimes P) \circ (S \otimes \rho 1') \circ (\rho' \otimes S) = \rho' \circ (P \otimes \mu), \tag{60}$$

$$\rho(p \otimes 1) = 1 \otimes p, \quad \textit{for all } p \in P. \tag{61}$$

If M is a right S-module, then $M \otimes_R RP$ possesses a structure of a right S-module given by

$$(m \otimes p) * s = \sum_{i=1}^{n} m s_i \otimes p_i,$$

where $\rho'(p\otimes s)=\sum_{i=1}^{n} s_i\otimes p_i$. *Moreover, if* M *is a unital* S*-bimodule, then* $M\otimes_R P$ *is a unital* S*-bimodule, where the structure of the right* S*-module is given by the structure of* M*.*

Example 5.12. let P be a central $\mathcal{Z}$-bimodule. Since R is a central $\mathcal{Z}$-bimodule too, we have a $\mathcal{Z}$-bimodule isomorphism $\rho : R\otimes_{\mathcal{Z}} P \longrightarrow P\otimes_{\mathcal{Z}} R$ defined by $\rho(r\otimes p)=p\otimes r$. It is easy to see that ρ satisfies the conditions of Lemma 5.11. Moreover, for any R-bimodule M, by Lemma 5.11, the tensor product $P\otimes_{\mathcal{Z}} M$ has a structure of an R-bimodule defined by

$$r_1 * (p\otimes m) * r_2 = p\otimes r_1 m r_2,$$

for all $p\in P, m\in M$ and $r_1,r_2\in R$.

Lemma 5.13. *Let* $R\subseteq S$ *be a ring extension with the same unity,* P *a unital* R*-bimodule and* $\rho : S\otimes_R P\longrightarrow P\otimes_R S$ *an* R*-bimodule homomorphism which satisfies conditions (58) and (59). If* M *is an* S*-bimodule, then there exists an* S*-bimodule isomorphism*

$$\begin{array}{rccc} \eta: & (P\otimes_R S)\otimes_S M & \longrightarrow & P\otimes_R M \\ & (p\otimes_R s)\otimes_S m & \longmapsto & p\otimes_R sm \end{array},$$

where $P\otimes_R S$ *and* $P\otimes_R M$ *are* S*-bimodules with the actions given in Lemma 5.11.*

Proof. Clearly, η is a well-defined right S-linear map. Given, $s,s'\in S$, $p\in P$ and $m\in M$, denote $\rho(s'\otimes_R p)=\sum_{i=1}^{n} p_i\otimes_R s'_i$. Then

$$\begin{aligned} \eta(s' * ((p\otimes_R s)\otimes_S m)) &= \eta\left(\sum_{i=1}^{n} p_i\otimes_R s'_i s\otimes_S m\right) = \sum_{i=1}^{n} p_i\otimes_R (s'_i s)m \\ &= \sum_{i=1}^{n} p_i\otimes_R s'_i(sm) = s' * (p\otimes_R sm) = s'\eta((p\otimes_R s)\otimes_S m). \end{aligned}$$

Hence η is S-bilinear. Its inverse is given by:

$$\begin{array}{rccc} \eta^{-1}: & P\otimes_R M & \longrightarrow & (P\otimes_R S)\otimes_S M, \\ & p\otimes_R m & \longmapsto & p\otimes_R 1\otimes_S m. \end{array}$$

Indeed, if $p\in P, m\in M$ and $s\in S$, then

$$\eta^{-1}(\eta(p\otimes_R s\otimes_S m)) = \eta^{-1}(p\otimes_R sm) = p\otimes_R 1\otimes_S sm = p\otimes_R s\otimes_S m.$$

On the other hand,

$$\eta(\eta^{-1}(p\otimes_R m)) = \eta(p\otimes_R 1\otimes_S m) = p\otimes_R 1m = p\otimes_R m.$$

Consequently, η is an S-bimodule isomorphism. □

Lemma 5.14. *Let* $R\subseteq S$ *be an extension of rings with the same unity, let* $[P]\in \mathbf{Pic}(R)$ *and suppose that* $\rho : S\otimes_R P\longrightarrow P\otimes_R S$ *is an* R*-bimodule isomorphism, which satisfies the conditions of Lemma 5.11. Denote by* $[P^{-1}]$ *the inverse of* $[P]$ *in* $\mathbf{Pic}(R)$ *and let* $P\otimes_R P^{-1}\xrightarrow{\mathfrak{l}} R \xleftarrow{\mathfrak{r}} P^{-1}\otimes_R P$ *be isomorphisms of* R*-bimodules.*

(i) *Let* $\overline{\rho} : S \otimes_R P^{-1} \longrightarrow P^{-1} \otimes_R S$ *be the* R*-bimodule isomorphism induced by* $\overline{\rho} = P^{-1} \otimes \rho^{-1} \otimes P^{-1}$*, that is,*

$$\begin{tikzcd}
S \otimes_R P^{-1} \arrow[r] \arrow[rrdd, dashed, bend right, "\overline{\rho}"'] & R \otimes_R S \otimes_R P^{-1} \arrow[r] & P^{-1} \otimes_R P \otimes_R S \otimes_R P^{-1} \arrow[d, "P^{-1}\otimes\rho^{-1}\otimes P^{-1}"] \\
 & & P^{-1} \otimes_R S \otimes_R P \otimes_R P^{-1} \arrow[d] \\
 & & P^{-1} \otimes_R S.
\end{tikzcd}$$

Suppose that $\overline{\rho}$ *satisfies the conditions of Lemma 5.11 and endow* $P^{-1} \otimes_R S$ *with the structure of an* S*-bimodule determined by* $\overline{\rho}$ *as in Lemma 5.11. Let* $\Psi : P \otimes_R (P^{-1} \otimes_R S) \longrightarrow S$ *be the isomorphism defined by*

$$\Psi : P \otimes_R (P^{-1} \otimes_R S) \xrightarrow{P\otimes\overline{\rho}^{-1}} P \otimes_R (S \otimes_R P^{-1}) \xrightarrow{\rho^{-1}\otimes P^{-1}} S \otimes_R P \otimes P^{-1} \xrightarrow{S\otimes\mathfrak{l}} S,$$

that is, $\Psi = (S \otimes \mathfrak{l}) \circ (\rho^{-1} \otimes P^{-1}) \circ (P \otimes \overline{\rho}^{-1})$*. Then*

$$\Psi(p \otimes \bar{p} \otimes s) = \mathfrak{l}(p \otimes \bar{p})s,$$

for all $s \in S$*,* $p \in P$ *and* $\bar{p} \in P^{-1}$*.*

(ii) *Let* $s \in S$*,* $p \in P$ *and* $\bar{p} \in P^{-1}$*. Write* $\sum_{k=1}^{l} \mathfrak{l}(p_k \otimes \bar{p}_k) = 1$*, and denote* $\rho(s \otimes p) = \sum_{i=1}^{n} p_i \otimes s_i$ *and* $\rho^{-1}(p_k \otimes s_i) = \sum_{j=1}^{m} s_{i,j} \otimes \bar{p}_{k,j}$*. Then*

$$s\mathfrak{l}(p \otimes \bar{p}) = \sum_{i,j,k} \mathfrak{l}(p_i \otimes \bar{p}_k)s_{i,j}\mathfrak{l}(p_{k,j} \otimes \bar{p}). \tag{62}$$

(iii) Ψ *is an isomorphism of* S*-bimodules.*
(iv) $[P \otimes_R S] \in \mathbf{Pic}(S)$*.*

Proof. Let $s \in S$, $\bar{p} \in P^{-1}$ and take $p_l \in P$, $\bar{p}_l \in P^{-1}$, $l = 1, 2, ..., n$, such that $\sum_{l=1}^{n} \mathfrak{r}(\bar{p}_l \otimes p_l) = 1$. Denoting, $\rho^{-1}(p_l \otimes s) = \sum_{i=1}^{m} s_i \otimes p_{l,i}$, we have

$$\overline{\rho}(s \otimes \bar{p}) = \sum_{l,i} \bar{p}_l \otimes s_i\mathfrak{l}(p_{l,i} \otimes \bar{p}).$$

Its inverse is given by

$$\overline{\rho}^{-1}(\bar{p} \otimes s) = \sum_{k,j} \mathfrak{r}(\bar{p} \otimes p_{k,j})s_j \otimes \bar{p}_k,$$

where $\sum_{k=1}^{l} \mathfrak{l}(p_k \otimes \bar{p}_k) = 1$ and $\rho(s \otimes p_k) = \sum_{j=1}^{m} p_{k,j} \otimes s_j$.

(i) Let $s \in S$, $p \in P$ and $\bar{p} \in P^{-1}$. Write $\sum\limits_{k=1}^{n} \mathfrak{l}(p_k \otimes \bar{p}_k) = 1$ and denote $\rho(s \otimes p_k) = \sum\limits_{j=1}^{m} p_{k,j} \otimes s_j$. Then

$$\begin{aligned} p \otimes \bar{p} \otimes s \xmapsto{P\otimes\bar{\rho}^{-1}} & \sum_{k,j} p \otimes \mathfrak{r}(\bar{p} \otimes p_{k,j}) s_j \otimes \bar{p}_k \xmapsto{\rho^{-1}\otimes P^{-1}} \sum_{k,j} \rho^{-1}(p\mathfrak{r}(\bar{p} \otimes p_{k,j}) \otimes s_j) \otimes \bar{p}_k \\ = \quad & \sum_{k,j} \rho^{-1}(\mathfrak{l}(p \otimes \bar{p}) p_{k,j} \otimes s_j) \otimes \bar{p}_k = \sum_{k,j} \mathfrak{l}(p \otimes \bar{p}) \rho^{-1}(p_{k,j} \otimes s_j) \otimes \bar{p}_k \\ = \quad & \sum_{k} \mathfrak{l}(p \otimes \bar{p}) s \otimes p_k \otimes \bar{p}_k \xmapsto{S\otimes\mathfrak{l}} \sum_{k} \mathfrak{l}(p \otimes \bar{p}) s \mathfrak{l}(p_k \otimes \bar{p}_k) \\ = \quad & \mathfrak{l}(p \otimes \bar{p}) s. \end{aligned}$$

Hence,

$$\Psi(p \otimes \bar{p} \otimes s) = (S \otimes \mathfrak{l}) \circ (\rho^{-1} \otimes P^{-1}) \circ (P \otimes \bar{\rho}^{-1})(p \otimes \bar{p} \otimes s) = \mathfrak{l}(p \otimes \bar{p}) s.$$

(ii) Let us compute $\Psi^{-1}(s\mathfrak{l}(p\otimes\bar{p}))$: observe that $\Psi^{-1} = (P\otimes\bar{\rho})\circ(\rho\otimes P^{-1})\circ(S\otimes\mathfrak{l}^{-1})$. If $\sum\limits_{k=1}^{n} \mathfrak{l}(p_k\otimes\bar{p}_k) = 1$, then

$$\begin{aligned} s\mathfrak{l}(p \otimes \bar{p}) \xmapsto{S\otimes\mathfrak{l}^{-1}} & \sum_{k=1}^{l} s\mathfrak{l}(p \otimes \bar{p}) \otimes p_k \otimes \bar{p}_k = \sum_{k=1}^{l} s \otimes \mathfrak{l}(p \otimes \bar{p}) p_k \otimes \bar{p}_k = \sum_{k=1}^{l} s \otimes p\mathfrak{r}(\bar{p} \otimes p_k) \otimes \bar{p}_k \\ = \quad & \sum_{k=1}^{l} s \otimes p \otimes \mathfrak{r}(\bar{p} \otimes p_k) \bar{p}_k = \sum_{k=1}^{l} s \otimes p \otimes \bar{p}\mathfrak{l}(p_k \otimes \bar{p}_k) = s \otimes p \otimes \bar{p} \\ \xmapsto{\rho\otimes P^{-1}} & \sum_{i=1}^{n} p_i \otimes s_i \otimes \bar{p} \xmapsto{P\otimes\bar{\rho}} \sum_{i,j,k} p_i \otimes \bar{p}_k \otimes s_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p}), \end{aligned}$$

that is, $\Psi^{-1}(s\mathfrak{l}(p \otimes \bar{p})) = \sum\limits_{i,j,k} p_i \otimes \bar{p}_k \otimes s_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p})$. Therefore,

$$s\mathfrak{l}(p \otimes \bar{p}) = \Psi\Psi^{-1}(s\mathfrak{l}(p \otimes \bar{p})) = \Psi\left(\sum_{i,j,k} p_i \otimes \bar{p}_k \otimes s_{i,j}\mathfrak{l}(p_{k,j} \otimes \bar{p})\right) = \sum_{i,j,k} \mathfrak{l}(p_i \otimes \bar{p}_k) s_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p}).$$

(iii) It is enough to verify that Ψ is left S-linear. Take $s, s' \in S$, $p \in P$ and $\bar{p} \in P^{-1}$. Let $p_k \in P$ and $\bar{p}_k \in P^{-1}$, be such that $\sum\limits_{k=1}^{l} \mathfrak{l}(p_k \otimes \bar{p}_k) = 1$ and denote $\rho(s' \otimes p) = \sum\limits_{i=1}^{n} p_i \otimes s'_i$ e $\rho^{-1}(p_k \otimes s'_i) = \sum\limits_{j=1}^{l} s'_{i,j} \otimes p_{k,j}$. Then $\bar{\rho}(s'_i \otimes \bar{p}) = \sum\limits_{k,l} \bar{p}_k \otimes s'_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p})$. Thus,

$$\begin{aligned} \Psi(s' * (p \otimes (\bar{p} \otimes s))) &= \Psi\left(\sum_{i=1}^{n} p_i \otimes s'_i * (\bar{p} \otimes s)\right) = \Psi\left(\sum_{i,j,k} p_i \otimes \bar{p}_k \otimes s'_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p}) s\right) \\ &= \sum_{i,j,k} \mathfrak{l}(p_i \otimes \bar{p}_k) s'_{i,j} \mathfrak{l}(p_{k,j} \otimes \bar{p}) s \overset{(62)}{=} s' \mathfrak{l}(p \otimes \bar{p}) s = s' \Psi(p \otimes \bar{p} \otimes s). \end{aligned}$$

(iv) By Lemma 5.13 and by item (iii), we have the R-bimodule isomorphisms

$$(P \otimes_R S) \otimes_S (P^{-1} \otimes_R S) \simeq P \otimes_R (P^{-1} \otimes_R S) \simeq S.$$

Similarly, $(P^{-1} \otimes_R S) \otimes_S (P \otimes_R S) \simeq S$, as S-bimodules. Consequently, $[P \otimes_R S] \in \mathbf{Pic}(S)$. $\square$

Theorem 5.15. *The sequence*

$$\mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)} \xrightarrow{\varphi_2} \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \xrightarrow{\varphi_3} \mathcal{C}_0(\Theta/R)$$

is exact.

Proof. If $[Q] \in \mathrm{Im}(\varphi_2)$, then there exists $[Q] \overset{[\phi]}{\Longrightarrow} [X] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)}$. By the proof of Proposition 5.3 there exists an R-bimodule isomorphism $g_x : \Theta_x \otimes Q \otimes \Theta_{x^{-1}} \longrightarrow Q \otimes R1_x$ determined by

$$g_x(u_x \otimes q \otimes u_{x^{-1}}) = q' \otimes 1_x, \text{ where } \phi(q') = u_x \phi(q) u_{x^{-1}}.$$

Let $h_x : Q \otimes \Theta_x \longrightarrow \Theta_x \otimes Q$ be an R-bimodule isomorphism, such that the following diagram is commutative:

$$\begin{tikzcd}
R1_x \otimes Q \otimes \Theta_x \arrow[r] & \Theta_x \otimes \Theta_{x^{-1}} \otimes Q \otimes \Theta_x \arrow[r, "\Theta_x \otimes g_{x^{-1}}"] & \Theta_x \otimes Q \otimes R1_{x^{-1}} \arrow[d] \\
Q \otimes R1_x \otimes \Theta_x \arrow[u] & & \Theta_x \otimes R1_{x^{-1}} \otimes Q \arrow[d] \\
Q \otimes \Theta_x \arrow[u] \arrow[rr, dashed, "h_x"'] & & \Theta_x \otimes Q
\end{tikzcd}$$

that is,

$$h_x(q \otimes u_x) = \sum_{(x)} \omega_x \otimes g_{x^{-1}}(\omega_{x^{-1}} \otimes q \otimes u_x),$$

where $1_x = \sum_{(x)} \omega_x \overset{\Theta}{\circ} \omega_{x^{-1}}$, with $\omega_x \in \Theta_x$, $\omega_{x^{-1}} \in \Theta_{x^{-1}}$, and arbitrary $q \in Q$, $u_x \in \Theta_x$.

Claim 5.16. *The isomorphisms h_x satisfy the commutative diagram:*

$$\begin{tikzcd}
\Theta_x \otimes Q \otimes \Theta_y \arrow[r, "h_x \otimes \Theta_y"] \arrow[d, "\Theta_x \otimes h_y"'] & Q \otimes \Theta_x \otimes \Theta_y \arrow[r, "Q \otimes f^{\Theta}_{x,y}"] & Q \otimes 1_x \Theta_{xy} \arrow[d, "1_x h_{xy}"] \\
\Theta_x \otimes \Theta_y \otimes Q \arrow[rr, "f^{\Theta}_{x,y} \otimes Q"'] & & 1_x \Theta_{xy} \otimes Q.
\end{tikzcd} \tag{63}$$

Indeed, let $1_x = \sum_{(x)} (\omega_x \overset{\Theta}{\circ} \omega_{x^{-1}})$ and $1_y = \sum_{(y)} (\omega_y \overset{\Theta}{\circ} \omega_{y^{-1}})$. By Remark 3.21, the restriction

$1_x h_{xy} : Q \otimes 1_x \Theta_{xy} \longrightarrow 1_x \Theta_{xy} \otimes Q$ is given by $1_x h_{xy}(q \otimes u_{xy}) = \sum_{(x),(y)} (\omega_x \overset{\Theta}{\circ} \omega_y) \otimes g_{(xy)^{-1}}((\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}}) \otimes q \otimes u_{xy})$, for all $u_{xy} \in 1_x \Theta_{xy}$. Thus, given $q \in Q$, $u_x \in \Theta_x$ and $u_y \in \Theta_y$, we have:

$$
\begin{aligned}
1_x h_{xy} \circ (Q \otimes f^{\Theta}_{x,y})(q \otimes u_x \otimes u_y) &= 1_x h_{xy}(q \otimes (u_x \overset{\Theta}{\circ} u_y)) \\
&= \sum_{(x),(y)} (\omega_x \overset{\Theta}{\circ} \omega_y) \otimes g_{(xy)^{-1}}((\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}}) \otimes q \otimes (u_x \overset{\Theta}{\circ} u_y)) \\
&= \sum_{(x),(y)} (\omega_x \overset{\Theta}{\circ} \omega_y) \otimes q_{x,y}, \qquad (64)
\end{aligned}
$$

where $\phi(q_{x,y}) = \sum_{(x),(y)} (\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}})\phi(q)(u_x \overset{\Theta}{\circ} u_y)$.

On the other hand, applying $(f^{\Theta}_{x,y} \otimes Q) \circ (\Theta_x \otimes h_y) \otimes (h_x \otimes \Theta_y)$, we obtain:

$$
\begin{aligned}
q \otimes u_x \otimes u_y \xmapsto{h_x \otimes \Theta_y} & \sum_{(x)} \omega_x \otimes g_x(\omega_{x^{-1}} \otimes q \otimes u_x) \otimes u_y \\
= \; & \sum_{(x)} \omega_x \otimes q'_x \otimes u_y, \quad \text{where } \phi(q'_x) = \sum_{(x)} \omega_{x^{-1}} \phi(q) u_x, \\
\xmapsto{\Theta_x \otimes h_y} & \sum_{(x),(y)} \omega_x \otimes \omega_y \otimes g_y(\omega_{y^{-1}} \otimes q'_x \otimes u_y) \\
= \; & \sum_{(x),(y)} \omega_x \otimes \omega_y \otimes q'_{x,y}, \\
\xmapsto{f^{\Theta}_{x,y} \otimes Q} & \sum_{(x),(y)} (\omega_x \overset{\Theta}{\circ} \omega_y) \otimes q'_{x,y}, \qquad (65)
\end{aligned}
$$

where

$$
\phi(q'_{x,y}) = \sum_{(x),(y)} \omega_{y^{-1}} \phi(q'_x) u_y = \sum_{(x),(y)} \omega_{y^{-1}} (\omega_{x^{-1}} \phi(q) u_x) u_y = \sum_{(x),(y)} (\omega_{y^{-1}} \overset{\Theta}{\circ} \omega_{x^{-1}}) \phi(q) (u_x \overset{\Theta}{\circ} u_y) = \phi(q_{x,y}).
$$

Since ϕ is injective, we get $q'_{x,y} = q_{x,y}$. Consequently, (64) and (65) imply that the diagram is commutative.

The commutativity of (63) may be expressed as follows: given $q \in Q, u_x \in \Theta_x$ and $u_y \in \Theta_y$, denote

$$
h_x(q \otimes u_x) = \sum_{i=1}^{n} u_x^i \otimes q_i \quad \text{and} \quad h_y(q_i \otimes u_y) = \sum_{j=1}^{m} u_y^{i,j} \otimes q_{i,j}.
$$

Then,

$$
h_{xy}(q \otimes (u_x \overset{\Theta}{\circ} u_y)) = \sum_{i,j} (u_x^i \overset{\Theta}{\circ} u_y^{i,j}) \otimes q_{i,j}. \qquad (66)
$$

Consider now the R-bimodule isomorphism $F_x : Q \otimes \Theta_x \otimes Q^{-1} \longrightarrow \Theta_x$ defined by $F_x := h_x \otimes Q^{-1}$, that is, given $q \in Q, u_x \in \Theta_x$ and $\bar{q} \in Q^{-1}$, if we denote $h_x(q \otimes u_x) = \sum_{i=1}^{n} u_x^i \otimes q_i$, then

$$
F_x(q \otimes u_x \otimes \bar{q}) = \sum_{i=1}^{n} u_x^i \mathfrak{l}(q_i \otimes \bar{q}) \in Q,
$$

where $f_x(q \otimes u_x \otimes \bar{q}) = \sum_{i=1}^{n} u_x^i \otimes q_i$, where $Q \otimes_R Q^{-1} \xrightarrow{\mathfrak{l}} R \xleftarrow{\mathfrak{r}} Q^{-1} \otimes_R Q$ are R-bimodule isomorphisms.

Claim 5.17. *$\varphi_3([Q]) = [\Delta(\Theta)]$ in $\mathcal{C}_0(\Theta/R)$.*

Indeed, it suffices to verify that the following diagram is commutative

$$\begin{tikzcd}
Q\otimes\Theta_x\otimes Q^{-1}\otimes Q\otimes\Theta_y\otimes Q^{-1} \arrow[r, "f^Q_{x,y}"] \arrow[d, "F_x\otimes F_y"'] & R1_x\otimes Q\otimes\Theta_{xy}\otimes Q^{-1} \arrow[d, "F_{xy}"] \\
\Theta_x\otimes\Theta_y \arrow[r, "f^\Theta_{x,y}"'] & R1_x\otimes\Theta_{xy}.
\end{tikzcd}$$

Let $q_1, q_2 \in Q, \bar{q}_1, \bar{q}_2 \in Q^{-1}, u_x \in \Theta_x$ e $u_y \in \Theta_y$. Write

$$h_x(q_1 \otimes u_x) = \sum_{i=1}^{n} u_x^i \otimes q_i \quad \text{and} \quad h_y(q_i \otimes \mathfrak{r}(\bar{q}_1 \otimes q_2)u_y) = \sum_{j=1}^{m} u_y^{i,j} \otimes q_{i,j}.$$

Then, by (66)

$$h_{xy}(q_i \otimes (u_x \overset{\Theta}{\circ} \mathfrak{r}(\bar{q}_1 \otimes q_2)u_y)) = \sum_{i,j} (u_x^i \overset{\Theta}{\circ} u_y^{i,j}) \otimes q_{i,j}.$$

Thus,

$$\begin{aligned}
(F_{xy} \circ f^Q_{x,y})(q_1 \otimes u_x \otimes \bar{q}_1 \otimes q_2 \otimes u_y \otimes \bar{q}_2) &= F_{xy}(q_1 \otimes (u_x \overset{\Theta}{\circ} \mathfrak{r}(\bar{q}_1 \otimes q_2)u_y) \otimes \bar{q}_2) \\
&= \sum_{i,j} (u_x^i \overset{\Theta}{\circ} u_y^{i,j})\mathfrak{l}(q_{i,j} \otimes \bar{q}_2).
\end{aligned}$$

On the other hand, applying $f^\Theta_{x,y} \circ (F_x \otimes F_y)$ to $q_1 \otimes u_x \otimes \bar{q}_1 \otimes q_2 \otimes u_y \otimes \bar{q}_2$, we obtain

$$(q_1 \otimes u_x \otimes \bar{q}_1 \otimes q_2 \otimes u_y \otimes \bar{q}_2) \mapsto \sum_{i=1}^{n} u_x^i \mathfrak{l}(q_i \otimes \bar{q}_1) \otimes h_y(q_2 \otimes u_y) \otimes \bar{q}_2$$

$$= \sum_{i=1}^{n} u_x^i \otimes \mathfrak{l}(q_i \otimes \bar{q}_1) h_y(q_2 \otimes u_y) \otimes \bar{q}_2 = \sum_{i=1}^{n} u_x^i \otimes h_y(\mathfrak{l}(q_i \otimes \bar{q}_1)q_2 \otimes u_y) \otimes \bar{q}_2$$

$$= \sum_{i=1}^{n} u_x^i \otimes h_y(q_i \mathfrak{r}(\bar{q}_1 \otimes q_2) \otimes u_y) \otimes \bar{q}_2 = \sum_{i=1}^{n} u_x^i \otimes h_y(q_i \otimes \mathfrak{r}(\bar{q}_1 \otimes q_2)u_y) \otimes \bar{q}_2$$

$$= \sum_{i,j} u_x^i \otimes u_y^{i,j} \otimes q_{i,j} \otimes \bar{q}_2 \mapsto \sum_{i,j} u_x^i \otimes u_y^{i,j} \mathfrak{l}(q_{i,j} \otimes \bar{q}_2)$$

$$\mapsto \sum_{i,j} (u_x^i \overset{\Theta}{\circ} u_y^{i,j})\mathfrak{l}(q_{i,j} \otimes \bar{q}_2).$$

Hence, $F_x \circ f^Q_{x,y} = f^\Theta_{x,y} \circ (F_x \otimes F_y)$, for all $x, y \in G$. Therefore, $\Delta(\Omega^Q) \simeq \Delta(\Theta)$ as generalized partial crossed products. This yields that $\varphi_3([Q]) = [\Delta(\Omega^Q)] = [\Delta(\Theta)]$ in $\mathcal{C}_0(\Theta/R)$ and, consequently, $[Q] \in \ker(\varphi_3)$.

If $[P] \in \ker(\varphi_3)$, then there exists an isomorphism of generalized partial crossed products $j : \Delta(\Omega^P) \longrightarrow \Delta(\Theta)$. This means that, for each $x \in G$, there exists an R-bimodule isomorphism $j_x : P \otimes \Theta_x \otimes P^{-1} \longrightarrow \Theta_x$ such that the diagram

$$\begin{tikzcd} P \otimes \Theta_x \otimes P^{-1} \otimes P \otimes \Theta_y \otimes P^{-1} \arrow[r, "f^P_{x,y}"] \arrow[d, "j_x \otimes j_y"'] & R1_x \otimes P \otimes \Theta_{xy} \otimes P^{-1} \arrow[d, "j_{xy}"] \\ \Theta_x \otimes \Theta_y \arrow[r, "f^{\Theta}_{x,y}"'] & R1_x \otimes \Theta_{xy} \end{tikzcd} \tag{67}$$

is commutative. Given $u_x \in \Theta_x$ e $u_y \in \Theta_y$, denote

$$j_x^{-1}(u_x) = \sum_{i=1}^{n} p_i \otimes u_x^i \otimes \bar{p}_i \ \text{ e } \ j_y^{-1}(u_y) = \sum_{j=1}^{m} p_j \otimes u_y^j \otimes \bar{p}_j. \tag{68}$$

Then, by the commutativity of the diagram, we obtain

$$j_{xy}^{-1}(u_x \overset{\Theta}{\circ} u_y) = \sum_{i,j} p_i \otimes (u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \overset{\Theta}{\circ} u_y^j) \otimes \bar{p}_j. \tag{69}$$

Let $i_x : \Theta_x \otimes P \longrightarrow P \otimes \Theta_x$ be defined by $i_x : \Theta_x \otimes P \xrightarrow{j_x^{-1} \otimes P} P \otimes \Theta_x \otimes P^{-1} \otimes P \longrightarrow P \otimes \Theta_x$, that is, given $p \in P$ and $u_x \in \Theta_x$, if $j_x^{-1}(u_x) = \sum_{i=1}^{n} p_i \otimes u_x^i \otimes \bar{p}_i$, then

$$i_x(u_x \otimes p) = \sum_{i=1}^{n} p_i \otimes u_x^i \mathfrak{r}(\bar{p}_i \otimes p).$$

Put $i := \bigoplus_{x \in G} i_x : \Delta(\Theta) \otimes P \longrightarrow P \otimes \Delta(\Theta)$. Let us verify that i satisfies conditions (58) and (59) of Lemma 5.11. The first condition is:

$$(P \otimes f^{\Theta}_{x,y}) \circ (i_x \otimes \Theta_y) \circ (\Theta_x \otimes i_y) = i_{xy} \circ (f^{\Theta}_{x,y} \otimes P) \ \ \forall\ x, y \in G.$$

Let $p \in P, u_x \in \Theta_x, u_y \in \Theta_y$ in the notation of (68), then $i_{xy} \circ (f^{\Theta}_{x,y} \otimes P)$ is given by:

$$\begin{aligned} u_x \otimes u_y \otimes p \ \mapsto\ & (u_x \overset{\Theta}{\circ} u_y) \otimes p \mapsto j_{xy}^{-1}(u_x \overset{\Theta}{\circ} u_y) \otimes p \\ \overset{(69)}{=} & \sum_{i,j} p_i \otimes \left(u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \overset{\Theta}{\circ} u_y^j\right) \otimes \bar{p}_j \otimes p \\ \mapsto & \sum_{i,j} p_i \otimes \left(u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \overset{\Theta}{\circ} u_y^j\right) \mathfrak{r}(\bar{p}_j \otimes p). \end{aligned}$$

On the other hand, $(P \otimes f^{\Theta}_{x,y}) \circ (i_x \otimes \Theta_y) \circ (\Theta_x \otimes i_y)$ is given by:

$$\begin{aligned} u_x \otimes u_y \otimes p \xmapsto{\Theta_x \otimes i_y} & \sum_{j=1}^{m} u_x \otimes p_j \otimes u_y^j \mathfrak{r}(\bar{p}_j \otimes p) \\ \xmapsto{i_x \otimes \Theta_y} & \sum_{i,j} p_i \otimes u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \otimes u_y^j \mathfrak{r}(\bar{p}_j \otimes p) \\ \xmapsto{P \otimes f^{\Theta}_{x,y}} & \sum_{i,j} p_i \otimes \left(u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \overset{\Theta}{\circ} u_y^j \mathfrak{r}(\bar{p}_j \otimes p)\right) \\ = \quad & \sum_{i,j} p_i \otimes \left(u_x^i \mathfrak{r}(\bar{p}_i \otimes p_j) \overset{\Theta}{\circ} u_y^j\right) \mathfrak{r}(\bar{p}_j \otimes p). \end{aligned}$$

Consequently, i satisfies condition (58). Since j is an isomorphism of generalized partial crossed products, then $j_1 \circ \nu_P = \iota$, where ν_P is given in Remark 5.9. Then

$$j_1^{-1}(1) = j_1^{-1}(\iota(1)) = \nu_P(1) = \sum_{k=1}^{n} p_k \otimes \iota(1) \otimes \bar{p}_k,$$

where $\sum_{k=1}^{n} \mathfrak{l}(p_k \otimes \bar{p}_k) = 1$, for all $r \in R$. Hence,

$$i_1(1 \otimes p) = \sum_{k=1}^{n} p_k \otimes \mathfrak{r}(\bar{p}_k \otimes p) = \sum_{k=1}^{n} p_k \mathfrak{r}(\bar{p}_k \otimes p) \otimes 1 = \sum_{k=1}^{n} \mathfrak{l}(p_k \otimes \bar{p}_k)p \otimes 1 = p \otimes 1.$$

Therefore, i satisfies also condition (59). It follows from Lemma 5.11 that $P \otimes \Delta(\Theta)$ is an $\Delta(\Theta)$-bimodule via:

$$(p \otimes u_x) * u_y = p \otimes (u_x \overset{\Theta}{\circ} u_y) \text{ and } u_y * (p \otimes u_x) = \sum_{i=1}^{n} p_i \otimes (u_y^i \overset{\Theta}{\circ} u_x),$$

where $i_y(u_y \otimes p) = \sum_{i=1}^{n} p_i \otimes u_y^i$. We construct the isomorphism $\overline{i_x} : \Theta_x \otimes P^{-1} \longrightarrow P^{-1} \otimes \Theta_x$ by:

$$\overline{i_x} : \Theta_x \otimes P^{-1} \longrightarrow P^{-1} \otimes P \otimes_R \Theta_x \otimes P^{-1} \xrightarrow{j_x} P^{-1} \otimes \Theta_x,$$

that is, $\overline{i_x}(u_x \otimes \bar{p}) = \sum_{k=1}^{n} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p})$, where $\sum_{k=1}^{n} \mathfrak{r}(\bar{p}_k \otimes p_k) = 1$. Let $\bar{i} = \bigoplus_{x \in G} \overline{i_x} : \Delta(\Theta) \otimes P^{-1} \longrightarrow P^{-1} \otimes_R \Delta(\Theta)$. We shall verify that $\bar{i}$ also satisfies the conditions of Lemma 5.11. Let $u_x \in \Theta_x, u_y \in \Theta_y$ and $\bar{p} \in P^{-1}$. Take any two decompositions $\sum_{k=1}^{n} \mathfrak{r}(\bar{p}_k \otimes p_k) = 1$ and $\sum_{l=1}^{m} \mathfrak{r}(\bar{p}_l \otimes p_l) = 1$. Then, $(P \otimes f_{x,y}^{\Theta}) \circ (\overline{i_x} \otimes \Theta_y) \circ (\Theta_x \otimes \overline{i_y})$ is given by:

$$\begin{aligned}
u_x \otimes u_y \otimes \bar{p} &\xmapsto{\Theta_x \otimes \overline{i_y}} \sum_{l=1}^{m} u_x \otimes \bar{p}_l \otimes j_y(p_l \otimes u_y \otimes \bar{p}) \\
&\xmapsto{\overline{i_x} \otimes \Theta_y} \sum_{k,l} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p}_l) \otimes j_y(p_l \otimes u_y \otimes \bar{p}) \\
&\xmapsto{P \otimes f_{x,y}^{\Theta}} \sum_{k,l} \bar{p}_k \otimes \left(j_x(p_k \otimes u_x \otimes \bar{p}_l) \overset{\Theta}{\circ} j_y(p_l \otimes u_y \otimes \bar{p})\right) \\
&\overset{(*)}{=} \sum_{k,l} \bar{p}_k \otimes j_{xy}(p_k \otimes (u_x \mathfrak{r}(\bar{p}_l \otimes p_l) \overset{\Theta}{\circ} u_y) \otimes \bar{p}) \\
&= \sum_{k=1}^{n} \bar{p}_k \otimes j_{xy}(p_k \otimes (u_x \overset{\Theta}{\circ} u_y) \otimes \bar{p}) \\
&= i_{xy}((u_x \overset{\Theta}{\circ} u_y) \otimes \bar{p}) \\
&= i_{xy} \circ (f_{x,y}^{\Theta} \otimes P^{-1})(u_x \otimes u_y \otimes \bar{p}),
\end{aligned}$$

where the equality $(*)$ comes from the commutativity of diagram (67). Consequently, $(P \otimes f^{\Theta}_{x,y}) \circ (\overline{i_x} \otimes \Theta_y) \circ (\Theta_x \otimes \overline{i_y}) = \overline{i_{xy}} \circ (f^{\Theta}_{x,y} \otimes P^{-1})$. For the second condition we have

$$\begin{aligned}\overline{i_1}(1 \otimes \bar{p}) &= \sum_{k=1}^{n} \bar{p}_k \otimes j_1(p_k \otimes 1 \otimes \bar{p}) = \sum_{k=1}^{n} \bar{p}_k \otimes \mathfrak{l}(p_k \otimes \bar{p}) = \sum_{k=1}^{n} \bar{p}_k \mathfrak{l}(p_k \otimes \bar{p}) \otimes 1 \\ &= \sum_{k=1}^{n} \mathfrak{r}(\bar{p}_k \otimes p_k)\bar{p} \otimes 1 = \bar{p} \otimes 1.\end{aligned}$$

Lemma 5.11 implies that $P^{-1} \otimes_R \Delta(\Theta)$ possesses a $\Delta(\Theta)$-bimodule structure given by $\bar{i}$.

Claim 5.18. *The isomorphism $\bar{i}$ is given by $\bar{i} = P^{-1} \otimes i^{-1} \otimes P^{-1}$, where $P^{-1} \otimes i^{-1} \otimes P^{-1} : \Delta(\Theta) \otimes P^{-1} \longrightarrow P^{-1} \otimes \Delta(\Theta)$ is defined as in item (i) of Lemma 5.14.*

Indeed, we first observe that $i_x^{-1} : P \otimes \Theta_x \longrightarrow \Theta_x \otimes P$ is given by $i_x^{-1}(p \otimes u_x) = \sum_{l=1}^{m} j_x(p \otimes u_x \otimes \bar{p}_l) \otimes p_l$, where $\sum_{l=1}^{m} \mathfrak{r}(\bar{p}_l \otimes p_l) = 1$. Now, applying $P^{-1} \otimes i^{-1} \otimes P^{-1}$ for $u_x \otimes \bar{p}$ we obtain:

$$\begin{aligned}u_x \otimes \bar{p} &\mapsto \sum_{k=1}^{n} \bar{p}_k \otimes p_k \otimes u_x \otimes \bar{p} \mapsto \sum_{k=1}^{n} \bar{p}_k \otimes i^{-1}(p_k \otimes u_x) \otimes \bar{p} = \sum_{k,l} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p}_l) \otimes p_l \otimes \bar{p} \\ &\mapsto \sum_{k,l} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p}_l)\mathfrak{l}(p_l \otimes \bar{p}) = \sum_{k,l} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p}_l \mathfrak{l}(p_l \otimes \bar{p})) \\ &= \sum_{k,l} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \mathfrak{r}(\bar{p}_l \otimes p_l)\bar{p}) = \sum_{k} \bar{p}_k \otimes j_x(p_k \otimes u_x \otimes \bar{p}) = \overline{i_x}(u_x \otimes \bar{p}).\end{aligned}$$

By item (iv) of Lemma 5.14, we get that $[P \otimes \Delta(\Theta)] \in \mathbf{Pic}(\Delta(\Theta))$. Let us verify that $[P] \xRightarrow{[\phi]} [P \otimes \Delta(\Theta)] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)}$, where

$$\begin{array}{rccc}\phi: & P & \longrightarrow & P \otimes \Delta(\Theta), \\ & p & \longmapsto & p \otimes 1.\end{array}$$

Note that

$$\begin{array}{rccc}\bar{\phi}_r: & P \otimes_R \Delta(\Theta) & \longrightarrow & P \otimes \Delta(\Theta), \\ & p \otimes u_x & \longrightarrow & \phi(p) * u_x\end{array}$$

is the identity, because

$$\phi(p) * u_x = (p \otimes 1) * u_x = p \otimes u_x = p \otimes u_x.$$

Thus, $\bar{\phi}_r$ is an R-$\Delta(\Theta)$-bimodule isomorphism. By Remark 2.13, we have

$$[P] \xRightarrow{[\phi]} [P \otimes \Delta(\Theta)] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R).$$

Let us now verify that $\phi(P) * \Theta_x = \Theta_x * \phi(P)$, for all $x \in G$. Let $p \in P$ and $u_x \in \Theta_x$. Denoting $i_x(u_x \otimes p) = \sum_{i=1}^{n} p_i \otimes u_x^i$, we compute

$$u_x * \phi(p) = u_x * (p \otimes 1) = \sum_{i=1}^{n} p_i \otimes (u_x^i \overset{\Theta}{\circ} 1) = \sum_{i=1}^{n} p_i \otimes u_x^i = \sum_{i=1}^{n} (p_i \otimes 1) * u_x^i = \sum_{i=1}^{n} \phi(p_i) * u_x^i \in \phi(P) * \Theta_x.$$

Consequently, $\Theta_x * \phi(P) \subseteq \phi(P) * \Theta_x$. On the other hand, since $p \otimes u_x \in P \otimes_R \Delta(\Theta)$ and $i : \Delta(\Theta) \otimes_R P \longrightarrow P \otimes_R \Delta(\Theta)$ is an isomorphism, there exist $u_x^j \in \Theta_x$ and $p_j \in P$, $j = 1, 2, ..., m$, such that $i_x \left(\sum_{j=1}^{m} u_x^j \otimes p_j \right) = p \otimes u_x$. Then,

$$\phi(p) * u_x = p \otimes u_x = i_x \left(\sum_{j=1}^{m} u_x^j \otimes p_j \right) = \sum_{j=1}^{m} u_x^j * (p_j \otimes 1) = \sum_{j=1}^{m} u_x^j * \phi(p_j) \in \Theta_x * \phi(P).$$

Thus, $\phi(P) * \Theta_x \subseteq \Theta_x * \phi(P)$ and, consequently, we obtain the desired equality. Therefore,

$$[P] \overset{[\phi]}{\Longrightarrow} [P \otimes_R \Delta(\Theta)] \ \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)}.$$

Obviously, $\varphi_2(\ [P] \overset{[\phi]}{\Longrightarrow} [P \otimes \Delta(\Theta)] \ ) = [P]$ and, hence, $[P] \in \mathrm{Im}(\varphi_2)$. $\square$

5.3. The group $\mathcal{B}(\Theta/R)$ and the third exact sequence

Define the group $\mathcal{B}(\Theta/R)$ by the exact sequence

$$\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} \xrightarrow{\mathcal{L}} \mathcal{C}(\Theta/R) \longrightarrow \mathcal{B}(\Theta/R) \longrightarrow 1 \ ,$$

that is,

$$\mathcal{B}(\Theta/R) = \frac{\mathcal{C}(\Theta/R)}{\mathrm{Im}(\mathcal{L})}.$$

Notation: If $f : G_1 \longrightarrow G_2$ is a homomorphism of groups, we denote by f^c the co-kernel of f, that is, $f^c : G_2 \longrightarrow G_2/\mathrm{Im}(f)$.

Proposition 5.19. *The sequence*

$$\mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \xrightarrow{\varphi_3} \mathcal{C}_0(\Theta/R) \xrightarrow{\varphi_4} \mathcal{B}(\Theta/R),$$

where φ_4 is $\mathcal{L}^c$ restricted to $\mathcal{C}_0(\Theta/R)$, is exact.

Proof. Clearly, $\mathrm{Im}(\varphi_3) \subseteq \ker(\varphi_4)$. Let $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$ be such that $\varphi_4([\Delta(\Gamma)]) = [1]$ in $\mathcal{B}(\Theta/R)$. Then there exists $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ such that $\mathcal{L}([P]) = [\Delta(\Gamma)]$. In particular, $\Gamma_x \simeq P \otimes \Theta_x \otimes P^{-1}$, for all $x \in G$. Thus, we have that $\Theta_x \simeq \Gamma_x \simeq P \otimes \Theta_x \otimes P^{-1}$, for all $x \in G$. Then $\Theta_x \otimes P \simeq P \otimes \Theta_x$, for all $x \in G$. Hence, $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*}$ and $[\Delta(\Gamma)] \in \mathrm{Im}(\varphi_3)$. $\square$

5.4. The group $\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ and the fourth exact sequence

Define

$$\mathbf{PicS}_0(R) = \{[P] \in \mathbf{PicS}(R);\ P|R \text{ as bimodules}\}.$$

Lemma 5.20.

(i) $\mathbf{PicS}_0(R)$ *is commutative.*
(ii) $\mathbf{PicS}_0(R) \subseteq \mathbf{PicS}_{\mathcal{Z}}(R)$.
(iii) *If* $[P] \in \mathbf{PicS}_0(R)$, *then* $[\Theta_x \otimes P \otimes \Theta_{x^{-1}}] \in \mathbf{PicS}_0(R)$.
(iv) $\mathcal{U}(\mathbf{PicS}_0(R)) \subseteq \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$.

Proof. Item (i) directly follows from Proposition 2.10.

(ii) Let $[P] \in \mathbf{PicS}_0(R)$. By Remark 2.9, we obtain that P is a central $\mathcal{Z}$-bimodule. Consequently, $\mathbf{PicS}_0(R) \subseteq \mathbf{PicS}_{\mathcal{Z}}(R)$.

(iii) If $[P] \in \mathbf{PicS}_0(R)$, then $P|R$ and by the compatibility with the tensor product, $\Theta_x \otimes P \otimes \Theta_{x^{-1}} | \Theta_x \otimes \Theta_{x^{-1}} \simeq R1_x$. It follows by the transitivity that $\Theta_x \otimes P \otimes \Theta_{x^{-1}}|R$, for each $x \in G$. Therefore, $[\Theta_x \otimes P \otimes \Theta_{x^{-1}}] \in \mathbf{PicS}_0(R)$.

(iv) If $[P] \in \mathcal{U}(\mathbf{PicS}_0(R))$, then by (ii) we have $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)$. Since $P|R$ and $P^{-1}|R$, it follows that $P \otimes \Theta_x \otimes P^{-1} | \Theta_x$, for each $x \in G$. Hence, $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$. $\square$

Items (ii) and (iii) of Lemma 5.20 imply that we may restrict each isomorphism $\alpha^*_x : \mathbf{PicS}_{\mathcal{Z}}(R)[R1_{x^{-1}}] \longrightarrow \mathbf{PicS}_{\mathcal{Z}}(R)[R1_x]$, defined by $\alpha^*_x([P][R1_x]) = [\Theta_x \otimes P \otimes \Theta_{x^{-1}}]$, to the ideal $\mathbf{PicS}_0(R)[R1_{x^{-1}}]$ and obtain a partial action of G on $\mathbf{PicS}_0(R)$.

Given $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$, since $\Gamma_x|\Theta_x$, then $\Gamma_x \otimes \Theta_{x^{-1}}|R$, for all $x \in G$. Thus, $[\Gamma_x \otimes \Theta_{x^{-1}}] \in \mathbf{PicS}_0(R)$. Define $f^\Gamma : G \longrightarrow \mathbf{PicS}_0(R)$ by $f^\Gamma(x) = [\Gamma_x \otimes \Theta_{x^{-1}}]$, for all $x \in G$. We have that $f^\Gamma(x) \in \mathbf{PicS}_0(R)[R1_x]$ and

$$[\Gamma_x \otimes \Theta_{x^{-1}}][\Theta_x \otimes \Gamma_{x^{-1}}] = [\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Theta_x \otimes \Gamma_{x^{-1}}] = [\Gamma_x \otimes R1_{x^{-1}} \otimes \Gamma_{x^{-1}}] = [R1_x].$$

Consequently, $f^\Gamma(x) = [\Gamma_x \otimes_R \Theta_{x^{-1}}] \in \mathcal{U}(\mathbf{PicS}_0(R)[R1_x])$, for each $x \in G$. Moreover, given $x, y \in G$, we obtain by (iii) of Lemma 3.7 that

$$\begin{aligned}
\alpha^*_x(f^\Gamma(y)[R1_{x^{-1}}])f^\Gamma(xy)^{-1}f^\Gamma(x) &= [\Theta_x \otimes \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Theta_{x^{-1}}][\Theta_{xy} \otimes \Gamma_{(xy)^{-1}}][\Gamma_x \otimes \Theta_{x^{-1}}] \\
&= [\Theta_x \otimes \Gamma_y \otimes \Theta_{y^{-1}} \otimes \Theta_{x^{-1}} \otimes \Theta_{xy} \otimes \Gamma_{(xy)^{-1}} \otimes \Gamma_x \otimes \Theta_{x^{-1}}] \\
&= [\Theta_x \otimes \Gamma_y \otimes R1_{y^{-1}} \otimes R1_{(xy)^{-1}} \otimes \Gamma_{(xy)^{-1}} \otimes \Gamma_x \otimes \Theta_{x^{-1}}] \\
&= [\Theta_x \otimes \Gamma_y \otimes \Gamma_{(xy)^{-1}} \otimes \Gamma_x \otimes \Theta_{x^{-1}}] \\
&= [\Theta_x \otimes R1_y \otimes R1_{x^{-1}} \otimes \Theta_{x^{-1}}] \\
&= [R1_{xy} \otimes \Theta_x \otimes \Theta_{x^{-1}}] \\
&= [R1_{xy} \otimes R1_x] = [R1_x][R1_{xy}].
\end{aligned}$$

Hence, $f^\Gamma \in Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$, for all $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$.

Lemma 5.21. *The map*

$$\begin{aligned}
\zeta : \quad \mathcal{C}(\Theta/R) &\longrightarrow Z^1(G, \alpha^*, \mathbf{PicS}_0(R)), \\
[\Delta(\Gamma)] &\longmapsto f^\Gamma,
\end{aligned}$$

is a group homomorphism whose kernel is $\mathcal{C}_0(\Theta/R)$.

Proof. If $[\Delta(\Gamma)] = [\Delta(\Omega)]$ in $\mathcal{C}(\Theta/R)$, we have $\Gamma_x \simeq \Omega_x$, for all $x \in G$. Then, $\Gamma_x \otimes \Theta_{x^{-1}} \simeq \Omega_x \otimes \Theta_{x^{-1}}$, for all $x \in G$. Therefore,

$$f^{\Gamma}(x) = [\Gamma_x \otimes \Theta_{x^{-1}}] = [\Omega_x \otimes \Theta_{x^{-1}}] = f^{\Omega}(x), \quad \text{for all } x \in G.$$

Thus, $f^{\Gamma} = f^{\Omega}$ in $Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$ and, consequently, ζ is well-defined.

Let $[\Delta(\Gamma)], [\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$. Since $[\Delta(\Gamma)][\Delta(\Omega)] = \left[\bigoplus_{x\in G} \Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x\right]$ in $\mathcal{C}(\Theta/R)$, then

$$f^{\Gamma\Omega}(x) = [\Gamma_x \otimes \Theta_{x^{-1}} \otimes \Omega_x \otimes \Theta_{x^{-1}}] = [\Gamma_x \otimes \Theta_{x^{-1}}][\Omega_x \otimes \Theta_{x^{-1}}] = f^{\Gamma}(x) f^{\Omega}(x),$$

for each $x \in G$. Hence, $f^{\Gamma\Omega} = f^{\Gamma} f^{\Omega}$ in $Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$. Therefore, ζ is a group homomorphism.

It remains to verify that $\ker(\zeta) = \mathcal{C}_0(\Theta/R)$. If $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$, then $\Gamma_x \simeq \Theta_x$, for each $x \in G$. Hence,

$$f^{\Gamma}(x) = [\Gamma_x \otimes \Theta_{x^{-1}}] = [\Theta_x \otimes \Theta_{x^{-1}}] = [R1_x], \text{ for all } x \in G.$$

Therefore, $[\Delta(\Gamma)] \in \ker(\zeta)$. On the other hand, let $[\Delta(\Gamma)] \in \ker(\zeta)$. Then, $\Gamma_x \otimes \Theta_{x^{-1}} \simeq R1_x$, for each $x \in G$. Taking the tensor product with Θ_x, we obtain $\Gamma_x \otimes R1_{x^{-1}} \simeq R1_x \otimes \Theta_x$ and consequently $\Gamma_x \simeq \Theta_x$, for all $x \in G$. Thus, $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$. $\square$

Define the group $\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ by the exact sequence

$$\begin{tikzcd}
\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} \arrow[rr, dashed] \arrow[dr, "\mathcal{L}"'] & & Z^1(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r] & \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r] & 1 \\
& \mathcal{C}(\Theta/R) \arrow[ur, "\zeta"'] & & &
\end{tikzcd}$$

that is,

$$\overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) = \frac{Z^1(G, \alpha^*, \mathbf{PicS}_0(R))}{(\zeta \circ \mathcal{L})(\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)})}.$$

Remark 5.22. A natural question is the following: what is the relation between $B^1(G, \alpha^*, \mathbf{PicS}_0(R))$ and the image of $\zeta \circ \mathcal{L}$? The inclusion $B^1(G, \alpha^*, \mathbf{PicS}_0(R)) \subseteq \mathrm{Im}(\zeta \circ \mathcal{L})$ always holds.

Indeed, let $\rho \in B^1(G, \alpha^*, \mathbf{PicS}_0(R))$, then there exists $[P] \in \mathcal{U}(\mathbf{PicS}_0(R)) \subseteq \mathbf{Pic}_{\mathcal{Z}}(R)^{(G)}$ (see Lemma 5.20), such that

$$\begin{aligned}
\rho(x) &= \alpha^*_x([P][R1_{x^{-1}}])[P^{-1}] = [\Theta_x \otimes P \otimes \Theta_{x^{-1}}][P^{-1}] = [\Theta_x \otimes P \otimes \Theta_{x^{-1}} \otimes P^{-1}] \\
&= [P^{-1} \otimes \Theta_x \otimes P \otimes \Theta_{x^{-1}}] = \zeta(\mathcal{L}([P^{-1}]))(x),
\end{aligned}$$

for all $x \in G$. Hence, $B^1(G, \alpha^*, \mathbf{PicS}_0(R)) \subseteq \mathrm{Im}(\zeta \circ \mathcal{L})$, as desired.

As in [28, Proposition 4.9] define the homomorphism $\varphi_5 : \mathcal{B}(\Theta/R) \longrightarrow \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$ by means of the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{C}_0(\Theta/R) \arrow[d, hook] \arrow[dr, "\varphi_4"] & & \\
\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} \arrow[r, "\mathcal{L}"] & \mathcal{C}(\Theta/R) \arrow[r, "\mathcal{L}^c"] \arrow[d, "\zeta"'] & \mathcal{B}(\Theta/R) \arrow[r] \arrow[d, dashed, "\varphi_5"] & 1 \\
& Z^1(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r, "(\zeta\mathcal{L})^c"] & \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r] & 1
\end{tikzcd}$$

The proof of the next proposition follows the same steps of the [28, Proposition 4.9].

Proposition 5.23. *The sequence*

$$\mathcal{C}_0(\Theta/R) \xrightarrow{\varphi_4} \mathcal{B}(\Theta/R) \xrightarrow{\varphi_5} \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R))$$

is exact.

5.5. The fifth exact sequence

Let g be a normalized 1-cocycle in $Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$, that is,

$$\begin{array}{rccl} g: & G & \longrightarrow & \mathbf{PicS}_0(R), \\ & x & \longrightarrow & [\nabla_x], \end{array}$$

where $[\nabla_x] \in \mathcal{U}(\mathbf{PicS}_0(R)[R1_x])$, for all $x \in G$ and

$$\alpha_x^*(g_y[R1_{x^{-1}}])g_{xy}^{-1}g_x = [R1_x][R1_{xy}], \quad \text{for all } x, y \in G. \tag{70}$$

Observe that

$$\alpha_x^*(g_y[R1_{x^{-1}}])[\Theta_x] = [\Theta_x]g_y, \quad \text{for all } x, y \in G. \tag{71}$$

Indeed,

$$\begin{aligned} \alpha_x^*(g_y[R1_{x^{-1}}])[\Theta_x] &= [\Theta_x \otimes \nabla_y \otimes \Theta_{x^{-1}}][\Theta_x] = [\Theta_x \otimes \nabla_y \otimes R1_{x^{-1}}] \\ &= [\Theta_x \otimes R1_{x^{-1}} \otimes \nabla_y] = [\Theta_x \otimes \nabla_y] = [\Theta_x]g_y. \end{aligned}$$

Lemma 5.24. *Let g be a normalized element in $Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$. Then*

$$\begin{array}{rccl} U: & G & \longrightarrow & \mathbf{PicS}(R), \\ & x & \longmapsto & g_x[\Theta_x], \end{array}$$

is a unital partial representation with $U_x \otimes U_{x^{-1}} \simeq R1_x$ and $U_x|\Theta_x$, for all $x \in G$.

Proof. Since g is normalized, then $[U_1] = g_1[\Theta_1] = [R]$. Given $x, y \in G$, we have

$$\begin{aligned} [U_x][U_y][U_{y^{-1}}] &= g_x[\Theta_x]g_y[\Theta_y]g_{y^{-1}}[\Theta_{y^{-1}}] \overset{(71)}{=} g_x\alpha_x^*(g_y[R1_{x^{-1}}])[\Theta_x][\Theta_y]g_{y^{-1}}[\Theta_{y^{-1}}] \\ &\overset{(70)}{=} g_{xy}[R1_x][R1_{xy}][R1_x][\Theta_{xy}]g_{y^{-1}}[\Theta_{y^{-1}}] = g_{xy}[R1_x][\Theta_{xy}]g_{y^{-1}}[\Theta_{y^{-1}}] \\ &= g_{xy}[\Theta_{xy}][R1_{y^{-1}}]g_{y^{-1}}[\Theta_{y^{-1}}] = g_{xy}[\Theta_{xy}]g_{y^{-1}}[R1_{y^{-1}}][\Theta_{y^{-1}}] \\ &= g_{xy}[\Theta_{xy}]g_{y^{-1}}[\Theta_{y^{-1}}] = [U_{xy}][U_{y^{-1}}]. \end{aligned}$$

Analogously, $[U_{x^{-1}}][U_x][U_y] = [U_{x^{-1}}][U_{xy}]$ for all $x, y \in G$. Again, since g is normalized, we have:

$$[U_x][U_{x^{-1}}] = g_x[\Theta_x]g_{x^{-1}}[\Theta_{x^{-1}}] \overset{(71)}{=} g_x\alpha_x^*(g_{x^{-1}}[R1_{x^{-1}}])[\Theta_x][\Theta_{x^{-1}}] = [g_1][R1_x] = [R][R1_x] = [R1_x],$$

for all $x \in G$. Moreover, since $g_x = [\nabla_x] \in \mathbf{PicS}_0(R)$, then $\nabla_x|R$, for all $x \in G$. Hence, $U_x \simeq \nabla_x \otimes \Theta_x|\Theta_x$, for all $x \in G$. $\square$

Due to the fact that U is a unital partial representation, Lemma 3.7 implies that there exists a family of R-bimodule isomorphisms

$$f^g = \{f^g_{x,y} : U_x \otimes_R U_y \longrightarrow R1_x \otimes U_{xy} \simeq 1_x U_{xy},\ \forall\ x, y \in G\}. \tag{72}$$

As $[U_1] = [R]$, we may choose a family of representatives $\{U_x\}_{x\in G}$ with $U_1 = R$ and a family of R-bimodule isomorphisms f^g with $f^g_{x,1} : U_x \otimes R \longrightarrow U_x$ and $f^g_{1,x} : R \otimes U_x \longrightarrow U_x$ given by the left and right actions of R on U_x, respectively. Then the commutative diagrams in (35) are trivially satisfied. By Corollary 3.16 there exists a partial 3-cocycle $\widetilde{\beta^g_{-,-,-}}$ in $Z^3_\Theta(G, \alpha, \mathcal{Z})$ associated to f^g.

Lemma 5.25. *The map*

$$\begin{array}{rccc} \delta: & Z^1(G, \alpha^*, \mathbf{PicS}_0(R)) & \longrightarrow & H^3_\Theta(G, \alpha, \mathcal{Z}), \\ & g & \longmapsto & [\widetilde{\beta^g_{-,-,-}}], \end{array}$$

is a group homomorphism. Moreover,

$$\delta \circ \zeta = [1], \tag{73}$$

where ζ is the homomorphism defined in Lemma 5.21.

Proof. By Proposition 3.17, $[\widetilde{\beta^g_{-,-,-}}] \in H^3_\Theta(G, \alpha, \mathcal{Z})$ does not depend on the choice of the family of R-bimodules $\{U_x\}_{x\in G}$, nor on the choice of the family of the R-bimodule isomorphisms f^g and, thus, δ is well-defined. Let $g, h \in Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$. Denote $[U_x] = g_x[\Theta_x]$ and $[V_x] = h_x[\Theta_x]$, for all $x \in G$. Let

$$f^g_{x,y} : U_x \otimes U_y \longrightarrow 1_x U_{xy} \quad \text{and} \quad f^h_{x,y} : V_x \otimes V_y \longrightarrow 1_x V_{x,y} \quad x, y \in G,$$

be families of R-bimodule isomorphisms and let $\widetilde{\beta^g_{-,-,-}}$ and $\widetilde{\beta^h_{-,-,-}}$ be the associated 3-cocycles in $Z^3_\Theta(G, \alpha, \mathcal{Z})$. Let $[W_x] = g_x h_x[\Theta_x]$, for all $x \in G$. Then,

$$[W_x] = g_x h_x[\Theta_x] = g_x h_x[R1_x][\Theta_x] = g_x[R1_x]h_x[\Theta_x] = g_x[\Theta_x][\Theta_{x^{-1}}]h_x[\Theta_x] = [U_x \otimes \Theta_{x^{-1}} \otimes V_x],$$

for each $x \in G$. Let $f^{gh}_{x,y}$ be the R-bimodule isomorphisms defined by

$$\begin{tikzcd}
U_x \otimes \Theta_{x^{-1}} \otimes V_x \otimes U_y \otimes \Theta_{y^{-1}} \otimes V_y \arrow[r, "T"] \arrow[ddr, dashed, bend right, "f^{gh}_{x,y}"'] & U_x \otimes U_y \otimes \Theta_{y^{-1}} \otimes \Theta_{x^{-1}} \otimes V_x \otimes V_y \arrow[d, "f^g_{x,y} \otimes f^\Theta_{x,y} \otimes f^h_{x,y}"] \\
& 1_x U_{xy} \otimes 1_{y^{-1}}\Theta_{(xy)^{-1}} \otimes 1_x V_{xy} \arrow[d, equal] \\
& 1_x U_{xy} \otimes \Theta_{(xy)^{-1}} \otimes V_{xy}
\end{tikzcd}$$

where $T : \Theta_{x^{-1}} \otimes V_x \otimes U_y \otimes \Theta_{y^{-1}} \longrightarrow U_y \otimes \Theta_{y^{-1}} \otimes \Theta_{x^{-1}} \otimes V_x$ is the isomorphism from Proposition 2.10. By Lemma 3.18, we obtain that $\widetilde{\beta^{gh}_{-,-,-}} = \widetilde{\beta^g_{-,-,-}}\widetilde{\beta^h_{-,-,-}}$. Hence,

$$\delta(gh) = [\widetilde{\beta^{gh}_{-,-,-}}] = [\widetilde{\beta^g_{-,-,-}}\widetilde{\beta^h_{-,-,-}}] = [\widetilde{\beta^g_{-,-,-}}][\widetilde{\beta^h_{-,-,-}}] = \delta(g)\delta(h),$$

and, consequently, δ is group homomorphism. As to (73), take $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$ and denote $f^\Gamma = \zeta([\Delta(\Gamma)])$, that is, $f^\Gamma_x = [\Gamma_x][\Theta_{x^{-1}}]$, for all $x \in G$. Thus,

$$[U_x] := f^\Gamma_x[\Theta_x] = [\Gamma_x][\Theta_{x^{-1}}][\Theta_x] = [\Gamma_x][R1_{x^{-1}}] = [\Gamma_x], \quad \text{for all } x \in G.$$

Consider a family of R-bimodule isomorphisms $f^\Gamma = \{f^\Gamma_{x,y} : \Gamma_x \otimes \Gamma_y \longrightarrow 1_x\Gamma_{xy},\ x, y \in G\}$ which is a factor set for $\Delta(\Gamma)$. Then $\widetilde{\beta^{f^\Gamma}_{-,-,-}}$ is trivial. Therefore, $\delta(\zeta([\Delta(\Gamma)])) = [1]$ em $H^3_\Theta(G, \alpha, \mathcal{Z})$. $\square$

As in [28, Proposition 4.10] define $\varphi_6 : \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) \longrightarrow H^3_\Theta(G, \alpha, \mathcal{Z})$ via the commutative diagram:

$$\begin{tikzcd}
\mathbf{Pic}_{\mathcal{Z}}(R)^{(G)} \arrow[r, "\zeta\mathcal{L}"] & Z^1(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r, "(\zeta\mathcal{L})^c"] \arrow[d, "\delta"'] & \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) \arrow[r] \arrow[dl, dashed, "\varphi_6"] & 1 \\
& H^3_\Theta(G, \alpha, \mathcal{Z}). & &
\end{tikzcd}$$

Proposition 5.26. *The sequence*

$$\mathcal{B}(\Theta/R) \xrightarrow{\varphi_5} \overline{H^1}(G, \alpha^*, \mathbf{PicS}_0(R)) \xrightarrow{\varphi_6} H^3_\Theta(G, \alpha, \mathcal{Z})$$

is exact.

Proof. Let $h \in Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$ be such that $(\zeta\mathcal{L})^c(h) \in Im(\varphi_5)$. Then there exists $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$, with $\varphi_5(\mathcal{L}^c([\Delta(\Gamma)])) = (\zeta\mathcal{L})^c(h)$. In view of the commutativity of the diagram and thanks to (73), we have

$$\varphi_6((\zeta\mathcal{L})^c(h)) = \varphi_6(\varphi_5(\mathcal{L}^c([\Delta(\Gamma)]))) = (\delta \circ \zeta)([\Delta(\Gamma)]) = [1].$$

Hence, $Im(\varphi_5) \subseteq \ker(\varphi_6)$.

For the converse inclusion, take $h \in Z^1(G, \alpha^*, \mathbf{PicS}_0(R))$ with $\varphi_6((\zeta\mathcal{L})^c(h)) = [1]$ in $H^3_\Theta(G, \alpha, \mathcal{Z})$. By the commutativity of the diagram we get that $\delta(h) = [1]$ in $H^3_\Theta(G, \alpha, \mathcal{Z})$. Denoting $\delta(h) = [\beta_{-,-,-}] \in B^3_\Theta(G, \alpha, \mathcal{Z})$, we have that there exists $\sigma_{-,-} : G \times G \longrightarrow \mathcal{Z}$, with $\sigma_{x,y} \in \mathcal{U}(\mathcal{Z}1_x1_{xy})$, for all $x, y \in G$, and

$$\beta_{x,y,z} = \alpha_x(\sigma_{y,z}1_{x^{-1}})\sigma^{-1}_{xy,z}\sigma_{x,yz}\sigma^{-1}_{x,y},$$

for all $x, y, z \in G$. Define $[\Omega_x] := h_x[\Theta_x]$, for all $x \in G$ and consider the family of associated isomorphisms f^h as in (72). Then, by definition, we have that $\beta_{x,y,x} \circ f^h_{x,yz} \circ (\Omega_x \otimes f^h_{y,z}) = f^h_{xy,z} \circ (f^h_{x,y} \otimes \Omega_z)$, for all $x, y, z \in G$. By Lemma 5.24, we obtain that $\Omega : G \longrightarrow \mathbf{PicS}(R)$, defined by $[\Omega_x] = h_x[\Theta_x]$, is a unital partial representation with $\Omega_x \otimes \Omega_{x^{-1}} \simeq R1_x$ and $\Omega_x|\Theta_x$, for all $x \in G$. By Proposition 3.15, we have that

$$\begin{array}{rccl} \bar{f}^h_{x,y} : & \Omega_x \otimes \Omega_y & \longrightarrow & 1_x\Omega_{xy}, \\ & \omega_x \otimes \omega_y & \longmapsto & \sigma_{x,y} f^h_{x,y}(\omega_x \otimes \omega_y), \end{array}$$

is a factor set for Ω. Hence, $\Delta(\Omega)$ is a generalized partial crossed product with $[\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$. Moreover, $\zeta([\Delta(\Omega)]) = h$. Then, by the commutativity of the diagram, we obtain

$$\varphi_5(\mathcal{L}^c([\Delta(\Omega)])) = ((\zeta\mathcal{L})^c \circ \zeta)([\Delta(\Omega)]) = (\zeta\mathcal{L})^c(h).$$

Thus, $(\zeta\mathcal{L})^c(h) \in \mathrm{Im}(\varphi_5)$. Therefore, $\ker(\varphi_6) \subseteq \mathrm{Im}(\varphi_5)$ and, consequently, the sequence is exact. $\square$

Recalling that $\mathcal{C}_0(\Theta/R) \simeq H^2_\Theta(G,\alpha,\mathcal{Z})$, thanks to Proposition 4.4, we derive from Theorems 5.5 and 5.15 and from Propositions 5.19, 5.23 and 5.26 our main result:

Theorem 5.27. *The sequence*

$$1 \longrightarrow H^1_\Theta(G,\alpha,\mathcal{Z}) \xrightarrow{\varphi_1} \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(G)} \xrightarrow{\varphi_2} \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{PicS}_{\mathcal{Z}}(R)^{\alpha^*} \xrightarrow{\varphi_3} H^2_\Theta(G,\alpha,\mathcal{Z})$$

$$\xrightarrow{\varphi_4} \mathcal{B}(\Theta/R) \xrightarrow{\varphi_5} \overline{H}^1(G,\alpha^*,\mathbf{PicS}_0(R)) \xrightarrow{\varphi_6} H^3_\Theta(G,\alpha,\mathcal{Z})$$

is exact.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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